Linear Programming Via Elementary Matrices

Helen Wang
INTERMODULAR DESCRIPTION SHEET:

UMAP Unit 584

Linear Programming Via Elementary Matrices

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MATHEMATICAL FIELD:

Linear algebra

TARGET AUDIENCE:

Students studying matrix algebra

PREREQUISITE SKILLS:

Familiarity with matrices and matrix operations; knowledge of elementary row operations is helpful but not required

OUTPUT SKILLS:

1) Basic knowledge of linear programming.
2) Understanding of and ability to use, from a matrix point of view, the simplex method for solving linear programming problems.
3) Understanding of elementary row operations as multiplication by nonsingular elementary matrices.

ABSTRACT:

This module provides a short matrix-based introduction to the simplex method, enabling a student familiar with matrix algebra (but not necessarily with row reduction techniques) to learn how to use the simplex method to solve a linear programming problem. Several basic references are supplied for the reader interested in exploring the many applications of linear programming as well as other approaches to the simplex method.

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Elementary Matrices

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Table of Contents

1. Linear Programming Problems .................................. 1
   1.1. What They Are ........................................ 1
   1.2. Uses .................................................. 3
   1.3. A Nutrition Problem ................................... 3
2. Preparing to Solve the Nutrition Problem ...................... 5
   2.1. Slack Variables ...................................... 6
   2.2. Elementary Matrices and Elementary Row Operations ... 8
3. The Simplex Method ............................................. 8
   3.1. Motivation and Discussion ............................ 8
   3.2. Summary .............................................. 10
4. Further Discussion .............................................. 11
5. References .................................................... 12
6. Answers to Exercises ........................................... 13
MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS (UMAP) PROJECT

The goal of UMAP was to develop, through a community of users and
developers, a system of instructional modules in undergraduate mathematics
and its applications to be used to supplement existing courses and from
which complete courses may eventually be built.

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1. Linear Programming Problems

1.1. What They Are

A linear programming problem consists of optimizing (maximizing or minimizing) a linear function of one or more variables, subject to one or more linear constraints on the variables. For example,

Find the minimum value of
\[ c = 15 + x_1 - 3x_2 \]

subject to
\[ x_1 + x_2 \leq 4 \]
\[ x_1 \geq 0 \]
\[ x_2 \geq 0 \]

is a linear programming problem. We may cast the problem in geometric terms: We wish to find the minimum height of the surface \( f(x_1, x_2) = 15 + x_1 - 3x_2 \) above the triangle \( T \) bounded by \( x_1 + x_2 = 4, x_1 = 0, \) and \( x_2 = 0 \) (Figure 1). Hence, if we envision placing a ball on that portion of the plane \( z = 15 + x_1 - 3x_2 \) lying above the triangle \( T \), then problem (1) is to determine the lowest point where the ball will settle under gravity in the negative \( z \) direction (Figure 2). We observe that the minimum value of \( c \) is 3, which occurs at \( x_1 = 0, x_2 = 4 \). This solution also makes sense algebraically, since to minimize \( c = 15 + x_1 - 3x_2 \) we need to make \( x_1 \) as small as possible and \( x_2 \) as large as possible.

Exercises

Solve the following linear programming problems by geometric and/or algebraic inspection, if possible.
1. Find the maximum value of
\[ c = 15 + x_1 - 3x_2 \]

subject to \( x_1 + x_2 \leq 4 \)
\[ x_1 \geq 0 \]
\[ x_2 \geq 0. \]

2. Find the minimum value of
\[ c = 21 - 2x_1 + 5x_2 \]
Figure 1. Constraint set for problem (1).

Figure 2. The triangular portion of the plane $z = 15 + x_1 - 3x_2$ lying above the constraint triangle $T$.

subject to $x_1 + 3x_2 \leq 9$

$x_1 \geq 0$

$x_2 \geq 0$.

3. Find the maximum value of

$z = 21 - 2x_1 + 5x_2$

subject to $x_1 + 3x_2 \leq 9$

$x_1 \geq 0$

$x_2 \geq 0$. 
4. Find the minimum value of
   \[ c = 17 + 3x_1 + 5x_2 \]
   subject to \( x_1 \geq 0, \ x_2 \geq 0 \).

5. Find the maximum value of
   \[ c = 17 + 3x_1 + 5x_2 \]
   subject to \( x_1 \geq 0, \ x_2 \geq 0 \).

6. Find the minimum value of
   \[ c = 17 + 3x_1 + 5x_2 \]
   subject to \( x_1 + x_2 \leq -1 \)
   \( x_1 \geq 0 \)
   \( x_2 \geq 0 \).

1.2. Uses

"Linear programming problems have a wide variety of applications, particularly in industry, including minimizing a (linear) cost function or maximizing a (linear) profit function, subject to (linear) constraints, such as warehouse capacities, budget constraints, or work force schedules. For an excellent historical introduction including applications, see [Dantzig 1963, Chapters 1, 2, 3, 15 and 17]."

In the next sections we will examine carefully one specific application of a linear programming problem. We will solve a linear programming maximization problem, concentrating on matrix algebra techniques rather than on geometric methods.

1.3. A Nutrition Problem

This problem consists of determining the amounts of various foods to eat to maximize the total nutrient (e.g., protein) content in one's diet, subject to budget and calorie constraints. Let's assume there are three foods in one's daily diet, and let \( x_1, x_2, \) and \( x_3 \) denote the number of ounces of each. Suppose there are 10 units of protein per ounce of food 1, 7 units of protein per ounce of food 2, and 8 units of protein per ounce of food 3. Then the total daily amount of protein from these foods is
   \[ c = 10x_1 + 7x_2 + 8x_3. \]
We wish to maximize $c$, subject to budget and calorie constraints. If our daily budget is $1.20, and food 1 costs $0.20 per ounce, food 2 costs $0.10 per ounce, and food 3 costs $0.30 per ounce, then the daily budget constraint is

$$(0.2)x_1 + (0.1)x_2 + (0.3)x_3 \leq 1.2,$$

or

$$2x_1 + x_2 + 3x_3 \leq 12.$$

Also, suppose we wish to restrict the daily calorie intake in the diet to no more than 1500 calories. If food 1 contains 200 calories per ounce, food 2 contains 200 calories per ounce and food 3 contains 100 calories per ounce, then this constraint is

$$200x_1 + 200x_2 + 100x_3 \leq 1500,$$

or

$$2x_1 + 2x_2 + x_3 \leq 15.$$

Of course, the number of units of each food must be nonnegative, so our problem is

Find the maximum value of

$$c = 10x_1 + 7x_2 + 8x_3$$

subject to

$$2x_1 + x_2 + 3x_3 \leq 12$$

$$2x_1 + 2x_2 + x_3 \leq 15$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Here geometric and/or algebraic inspection does not readily yield an answer; we will learn a technique called the simplex method for finding a solution.

**Exercise**

7. Write the following problem as a linear programming problem:

A bakery sells two kinds of cookies, sugar cookies and gingerbread cookies. It takes 1/2 hour to make a pound of sugar cookies, 1 hour to make a pound of gingerbread cookies, and the bakery operates 12 hours per day. Only one type of cookie can be made at a time. Also, the bakery is able to sell no more than 20 pounds of cookies per day. If the bakery makes a profit of 30¢ per pound on the sugar cookies and 50¢ per pound on the gingerbread
cookies, how many pounds of each should the bakery make each
day in order to maximize profits? (Do not solve yet. See Exercise
19.)

2. Preparing to Solve
the Nutrition Problem

2.1. Slack Variables

Before we introduce the simplex method for solving linear
programming problems, it will be helpful to be able to convert all
constraints to either equality constraints or to nonnegativity con-
straints. This can be done by introducing new variables to take up
the "slack" in each inequality constraint; hence the name slack
variable. For example, consider the inequality constraint

\[ 2x_1 + x_2 + 3x_3 \leq 12. \] (3)

We introduce a new nonnegative variable \( x_4 \) and consider the
equality constraint

\[ 2x_1 + x_2 + 3x_3 + x_4 = 12. \] (4)

If \( x_1, x_2, \) and \( x_3 \) satisfy (3), then (4) is satisfied with
\[ x_4 = 12 - (2x_1 + x_2 + 3x_3) \geq 0. \]

Conversely, if \( x_1, x_2, x_3 \) and \( x_4 \) satisfy (4) with \( x_4 \geq 0 \), then \( x_1, x_2, \) and \( x_3 \) satisfy (3). Similarly, we can replace the inequality constraint
\[ 2x_1 + 2x_2 + x_3 + x_4 \leq 15 \]
with the equality constraint
\[ 2x_1 + 2x_2 + x_3 + x_4 = 15, \]
where the new slack variable \( x_5 \) must be nonnegative.
Consequently, by using two nonnegative slack variables \( x_4 \) and \( x_5 \),
the original nutrition problem (2), which had two inequality con-
straints and three nonnegative variables, becomes

Find the maximum value of
\[ c = 10x_1 + 7x_2 + 8x_3 \]
subject to
\[ \begin{align*}
2x_1 + x_2 + 3x_3 + x_4 & = 12 \\
2x_1 + 2x_2 + x_3 + x_5 & = 15 \\
x_1, x_2, x_3, x_4, x_5 & \geq 0,
\end{align*} \] (5)

with two equality constraints and five nonnegative variables.
We are almost ready to apply the simplex method of solution to form (5) of the nutrition problem; we will complete our preparation with the introduction in the next section of two special square matrices called elementary matrices.

Exercises
Introduce appropriate slack variables to change the following inequality constraints to equality constraints with nonnegative constraints.
8. \(3x_1 + x_2 \leq 7\).

9. \(x_1 + 7x_2 \geq 5\).

### 2.2. Elementary Matrices and Elementary Row Operations

We begin by writing problem (3) in matrix notation.

Maximize \(c\) subject to

\[
\begin{align*}
\begin{bmatrix}
2 & 1 & 3 & 1 & 0 \\
2 & 2 & 1 & 0 & 1 \\
10 & 7 & 8 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
&=
\begin{bmatrix}
12 \\
15 \\
c
\end{bmatrix}
\end{align*}
\]

(6)

and \(x_1, x_2, x_3, x_4, x_5 \geq 0\).

Our goal is to rewrite (6) in an equivalent form that is easier to solve. We will use a matrix technique. (If you have learned to solve systems of linear equations by row reduction, the techniques which follow may remind you of elementary row operations and the Gauss-Jordan procedure. However, such knowledge is not necessary here.)

First, if we left-multiply both sides of the matrix equation in (6) by the same nonsingular \(3 \times 3\) matrix, then the resulting matrix equation will be equivalent to the original one. (Recall that a square matrix is nonsingular, or invertible, if it has an inverse under matrix multiplication.) Consider the effect of multiplying both sides of the matrix equation in (6) on the left by the matrix

\[
\begin{bmatrix}
1/2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

This operation replaces row 1 of the matrix on each side of (6) with
1/2 of row 1 (see Exercise 10):

\[
\begin{bmatrix}
1 & 1/2 & 3/2 & 1/2 & 0 \\
2 & 2 & 1 & 0 & 1 \\
10 & 7 & 8 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
= 
\begin{bmatrix}
6 \\
15 \\
c
\end{bmatrix}
\tag{7}
\]

In general, if \( A(i, r) \) is the \( 3 \times 3 \) nonsingular matrix that is the \( 3 \times 3 \) identity matrix with the \( i \)th diagonal element replaced by the nonzero constant \( r \), then left-multiplication by \( A(i, r) \) of a \( 3 \times n \) matrix \( (n \) can be any positive integer) replaces row \( i \) of that matrix with \( r \) times row \( i \) (see Exercises 11–13).

Now we will determine the effect of left-multiplying both sides of the matrix equation in (7) by

\[
\begin{bmatrix}
1 & 0 & 0 \\
-2 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

This replaces row 2 of the matrix on each side of (7) with row 2 plus \(-2\) times row 1 (see Exercise 14):

\[
\begin{bmatrix}
1 & 1/2 & 3/2 & 1/2 & 0 \\
0 & 1 & -2 & -1 & 1 \\
10 & 7 & 8 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
= 
\begin{bmatrix}
6 \\
3 \\
c
\end{bmatrix}
\tag{8}
\]

In general, if \( B(i, j, r) \) is the \( 3 \times 3 \) nonsingular matrix which is the \( 3 \times 3 \) identity matrix with the element in row \( i \) and column \( j \) \((i \neq j)\) replaced by \( r \), then left-multiplication by \( B(i, j, r) \) of a \( 3 \times n \) matrix \( (n \) can be any positive integer) replaces row \( i \) of that matrix with row \( i \) plus \( r \) times row \( j \) (see Exercises 15–17).

Matrices of the form \( A(i, r) \) and \( B(i, j, r) \) are \( 3 \times 3 \) elementary matrices, and left-multiplication by one of them is an elementary row operation. (See Exercises 11, 15, and 18.)

**Exercises**

10. Verify that equation (7) is the result of left-multiplying both sides of equation (6) by the matrix

\[
\begin{bmatrix}
1/2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

7
11. Verify that left-multiplication by \( A(i, r) \) of a \( 3 \times n \) matrix \( M \) replaces row \( i \) of \( M \) with \( r \) times row \( i \).

12. Verify that \( A(i, r) \) is a nonsingular \( 3 \times 3 \) matrix if \( r \neq 0 \). What is the inverse of \( A(i, r) \)?

13. How would you define a \( 4 \times 4 \) \( A(i, r) \) for replacing row \( i \) of a \( 4 \times n \) matrix \( M \) with \( r \) times row \( i \)? An \( m \times n \) matrix \( M \)?

14. Verify that equation (8) is the result of left-multiplying both sides of equation (7) by the matrix

\[
\begin{bmatrix}
  1 & 0 & 0 \\
  -2 & 1 & 0 \\
  0 & 0 & 1 
\end{bmatrix}
\]

15. Verify that left-multiplication by \( B(i, j, r) \) of a \( 3 \times n \) matrix \( M \) replaces row \( i \) of \( M \) with row \( i \) plus \( r \) times row \( j \) (\( i \neq j \)).

16. Verify that \( B(i, j, r) \) is a nonsingular \( 3 \times 3 \) matrix if \( i \neq j \). What is the inverse of \( B(i, j, r) \)?

17. How would you define a \( 4 \times 4 \) \( B(i, j, r) \) for replacing row \( i \) of a \( 4 \times n \) matrix \( M \) with row \( i \) plus \( r \) times row \( j \)? An \( m \times n \) matrix \( M \)?

18. Determine a nonsingular \( 3 \times 3 \) matrix \( C(i, j) \) such that left-multiplication by \( C(i, j) \) of a \( 3 \times n \) matrix \( M \) has the effect of interchanging rows \( i \) and \( j \) of \( M \). Verify that your \( C(i, j) \) is nonsingular; what is the inverse of \( C(i, j) \)? Extend your result to \( m \times n \) matrices \( M \).

\( (C(i, j) \) is the third of the three types of elementary matrix corresponding to an elementary row operation. We will not need it for the simplex method.)

3. The Simplex Method

3.1. Motivation and Discussion

Our goal is to left-multiply both sides of the matrix equation for the nutrition problem (6) by appropriate \( 3 \times 3 \) nonsingular elementary matrices, to obtain an equivalent form of (6) that is easier to solve.
I. We will start with some simple values for \(x_1, \ldots, x_5\) that satisfy (6) and the additional nonnegativity constraints \(x_1, \ldots, x_5 \geq 0\). The easiest values that fit all the constraints are the original variables set to zero and the slack variables set to the values on the right-hand side: \(x_1 = x_2 = x_3 = 0, \ x_4 = 12\), and \(x_5 = 15\). Then \(c\) has the value 0. However, \(c\) is not at its maximum value, since \(c\) would become positive if we let \(x_1, x_2, \) or \(x_3\) become positive. Of course, we would then have to reduce \(x_4\) from 12 and \(x_5\) from 15, in order to maintain the constraints \(2x_1 + x_2 + 3x_3 + x_4 = 12\) and \(2x_1 + 2x_2 + x_3 + x_5 = 15\).

II. In order to increase the value of \(c\), the largest increase per unit is obtained by letting \(x_1\) increase, since \(x_1\) increases \(c\) by 10 units per ounce, whereas \(x_2\) and \(x_3\) only increase \(c\) by 7 and 8 units per ounce, respectively. With \(x_2\) and \(x_3\) remaining at 0, row 1 of equation (6) says \(2x_1 + x_4 = 12\), and row 2 says \(2x_1 + x_5 = 15\). Since \(x_1 \geq 0\) and \(x_5 \geq 0\), row 1 requires \(x_1 \leq 12/2 = 6\) and row 2 requires \(x_1 \leq 15/2\). We want \(x_1\) to satisfy both restrictions, hence we must retain the stronger restriction \(x_1 \leq 6\) from row 1. The coefficient we wish to retain is the one in row 1 and column 1 of the left-hand matrix of (6). We will rewrite (6) in an equivalent form so column 1 of the left-hand matrix is replaced by the column vector \([1 \ 0 \ 0]^{T}\); this operation is called pivoting on the element in row 1 and column 1. We accomplish it by left-multiplying both sides of (6) by the product of elementary matrices

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-10 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
1/2 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1/2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

We obtain first (7) and then (8) of Section 2, and finally

\[
\begin{bmatrix}
1 & 1/2 & 3/2 & 1/2 & 0 \\
0 & 1 & -2 & -1 & 1 \\
0 & 2 & -7 & -5 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
= 
\begin{bmatrix}
6 \\
3 \\
\epsilon - 60
\end{bmatrix}.
\]

Note that first the column and then the row of the pivot value are chosen as follows: The column of the pivot value corresponds to the greatest rate of increase (for a maximization problem), and the row of the pivot value corresponds to the stronger restriction.

III. We now repeat the analysis of steps I and II on equation (9). Now the easiest nonnegative values to fit the constraints are \(x_2 = x_3 = x_4 = 0, \ x_1 = 6,\) and \(x_5 = 3,\) giving \(\epsilon = 60\). From row 3 of equation (9), \(\epsilon = 60 + 2x_2 - 7x_3 - 5x_4,\) hence \(\epsilon\) can still be in-
creased by letting $x_2$ increase. The restrictions on $x_2$ from rows 1 and 2 are $x_2 \leq 6/(1/2) = 12$ and $x_2 \leq 3$, respectively; hence we retain the stronger row 2 restriction on $x_2$. We now rewrite (9) in an equivalent form so column 2 of the left-hand matrix is replaced by the column vector $[0 \ 1 \ 0]^T$, i.e. we pivot on the element in row 2 and column 2. Therefore we left-multiply both sides of (9) by the product of elementary matrices

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & -1/2 \\
0 & 1 \\
0 & 0
\end{bmatrix}
$$

obtaining

$$
\begin{bmatrix}
1 & 0 & 5/2 & 1 & -1/2 \\
0 & 1 & -2 & -1 & 1 \\
0 & 0 & -3 & 3 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
= 
\begin{bmatrix}
9/2 \\
3 \\
\epsilon - 66
\end{bmatrix}
$$

(10)

IV. Again, we repeat the analysis of steps I and II on equation (10). The easiest nonnegative values to fit the constraints are $x_3 = x_4 = x_5 = 0$, $x_1 = 9/2$, and $x_2 = 3$, giving $\epsilon = 66$. From row 3 of (11), $\epsilon = 66 - 3x_3 - 3x_4 - 2x_5$; and since $x_3$, $x_4$ and $x_5$ must be $\geq 0$, $\epsilon$ cannot be increased further. Therefore $\epsilon = 66$ is the maximum number of daily protein units possible, achieved with $9/2$ ounces of food 1, 3 ounces of food 2, and 0 ounces of food 3.

3.2. Summary

The goal of the simplex method for a maximization problem is to rewrite the original matrix equation in an equivalent form such that the last row of the left-hand matrix contains only zeroes and negative numbers, and such that the last row of the right-hand matrix is $\epsilon$ minus a constant. Then the maximum value of $\epsilon$ can be readily obtained from the new equivalent matrix equation, as in step IV of Section 3.1. The simplex method for our problem consists of repeating the following two steps until the goal is achieved:

I. Find the simplest (all but two values 0, where two is the number of equality constraints) nonnegative values $x_1, \ldots, x_5$ to satisfy the matrix equation, and determine whether $\epsilon$ is already at its maximum.

II. If $\epsilon$ is not at its maximum, determine the appropriate pivot element. Choose the largest positive number in the last row of the
left-hand matrix, and let \( j \) denote its column number. Take the quotients of the elements of the rows of the right-hand matrix except for the last row, over the corresponding row elements in column \( j \) of the left-hand matrix; let \( i \) denote the row number that gives the smallest positive quotient. Pivot on the element in row \( i \) and column \( j \) (i.e., replace column \( j \) with 0's except for a 1 in row \( i \)) of the left-hand matrix via multiplication by appropriate elementary matrices, obtaining a new equivalent matrix equation. (Notice that we never pivot on an element in the last row.)

We have solved a problem of maximizing \( c \). In order to solve a problem of minimizing \( c \), we could first maximize \(-c\), and the minimum value of \( c \) would then be the negative of the maximum value of \(-c\).

Exercises
19. Use the simplex method to solve the linear programming problem of Exercise 7.

20. Use the simplex method to solve the following:

Find the maximum value of
\[
c = 5x_1 + 8x_2
\]
subject to
\[
2x_1 + 5x_2 \leq 40
\]
\[
x_1 + x_2 \leq 15
\]
\[
4x_1 + x_2 \leq 48
\]
\[
x_1, x_2 \geq 0.
\]

21. Several interesting applications of linear programming (both examples and exercises), which the reader can now solve, may be found in [Rorres and Anton 1984, Chapters 20, 22].

4. Further Discussion

We have provided a brief look at the simplex method for solving a linear programming problem. However, several important mathematical questions are beyond what we can present here. For example, we have to ask whether repeating steps I and II of Section 3.2 will always eventually lead us to a solution, as well as whether it is always possible to carry out these steps. We should also specify a more rigorous description of them.
In addition, there are variations on the simplex method, such as the revised simplex method and methods related to the dual problem [Luenberger 1973, Noble 1969]. The simplex method also has an interesting geometric interpretation related to convex sets [Luenberger 1973].

5. References


Chapter 1 presents a systematic description of matrix arithmetic, elementary matrices, and Gaussian elimination for solving systems of linear equations.


*Detailed introduction to linear programming and the simplex method, including origins, history, and applications.*


*Provides an excellent summary of a recent exciting development in linear programming, as well as a survey of linear programming in general.*


*Chapters 2 and 3 deal with basic properties of linear programming, convexity, the simplex method, and the revised simplex method. Chapter 4 discusses duality and the dual simplex method.*


*Chapter 6 contains an excellent rigorous matrix-based introduction to linear programming and the simplex method, as well as discussions of the role of convexity and the dual problem.*


*Chapters 20–22 provide an introduction to linear programming, including geometric interpretation, the simplex method, and applications.*


*These UMAP Modules treat linear programming graphically, with the second focusing on sensitivity analysis.*

Presents the classical “diet problem,” minimizing the cost of the diet subject to nutrition constraints.


Chapter 1 includes a presentation of Gaussian elimination and elementary matrices. Linear programming, including the simplex method and duality theory, is presented in Chapter 8.

6. **Answers to Exercises**

1. Maximum value of $\epsilon$ is 19 with $x_1 = 4$, $x_2 = 0$ (see Figure 2).

2. Minimum value of $\epsilon$ is 3 with $x_1 = 9$, $x_2 = 0$.

3. Maximum value of $\epsilon$ is 36 with $x_1 = 0$, $x_2 = 3$.

4. Minimum value of $\epsilon$ is 17 with $x_1 = x_2 = 0$.

5. $\epsilon$ has no maximum value; $\epsilon$ can be arbitrarily large as $x_1$ and $x_2$ become arbitrarily large.

6. There are no values for $x_1$ and $x_2$ which satisfy these constraints, since $x_1 \geq 0$ and $x_2 \geq 0$ imply $x_1 + x_2 \geq 0$. This problem is called *infeasible*.

7. Let $x_1$ denote the number of pounds of sugar cookies baked per day, and $x_2$ the number of pounds of gingerbread cookies. The time constraint is

\[
\frac{1}{2}x_1 + x_2 \leq 12, \quad \text{or} \quad x_1 + 2x_2 \leq 24.
\]

The total sales constraint is $x_1 + x_2 \leq 20$. We wish to maximize daily profit, which is $30x_1 + 50x_2$. Hence the problem is:

Find the maximum value of

\[
\epsilon = 30x_1 + 50x_2
\]

subject to

\[
x_1 + 2x_2 \leq 24
\]

\[
x_1 + x_2 \leq 20
\]

\[
x_1, x_2 \geq 0.
\]

(See Exercise 19 for the simplex method solution.)

8. $3x_1 + x_2 \leq 7$ is equivalent to $3x_1 + x_2 + x_3 = 7$ with $x_3 \geq 0$.

9. $x_1 + 7x_2 \geq 5$ is equivalent to $x_1 + 7x_2 - x_4 = 5$ with $x_4 \geq 0$. 

13
10. \[
\begin{bmatrix}
1/2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
2 & 1 & 3 & 1 \\
2 & 2 & 1 & 0 \\
10 & 7 & 8 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
1 & 1/2 & 3/2 & 1/2 & 0 \\
2 & 2 & 1 & 0 & 1 \\
10 & 7 & 8 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
12 \\
15 \\
7 \\
\end{bmatrix} = \begin{bmatrix}
6 \\
15 \\
i \\
\end{bmatrix}.
\]

11. Since \(A(i, r)\) is the identity matrix with the \(i\)th diagonal element replaced by \(r\), then all rows except the \(i\)th row of \(M\) stay the same after left multiplication by \(A(i, r)\), and each element of the \(i\)th row of \(M\) is multiplied by \(r\).

12. Since \(\det A(i, r) = r \neq 0\), then \(A(i, r)\) is nonsingular. In fact,
\[
A(i, r)^{-1} = A(i, 1/r) \quad \text{for } r \neq 0.
\]

13. For an \(m \times n\) matrix \(M\), let \(A(i, r)\) be the \(m \times m\) identity matrix with the \(i\)th diagonal element replaced with \(r\). Then left multiplication of \(M\) by \(A(i, r)\) replaces row \(i\) of \(M\) with \(r\) times row \(i\).

14. \[
\begin{bmatrix}
1 & 0 & 0 \\
-2 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1/2 & 3/2 & 1/2 & 0 \\
2 & 2 & 1 & 0 \\
10 & 7 & 8 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
1 & 1/2 & 3/2 & 1/2 & 0 \\
0 & 1 & -2 & 1 & 1 \\
10 & 7 & 8 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
6 \\
15 \\
i \\
\end{bmatrix} = \begin{bmatrix}
6 \\
i \\
\end{bmatrix}.
\]

15. Since \(B(i, j, r)\) is the identity matrix except in the \(i\)th row, then all rows except the \(i\)th row of \(M\) remain the same after left multiplication by \(B(i, j, r)\). Since \(i \neq j\), after left multiplication by \(B(i, j, r)\), row \(i\) of \(M\) becomes row \(i\) plus \(r\) times row \(j\); since \(B(i, j, r)\) has 1 in row \(i\) and column \(i\) and the constant \(r\) in row \(i\) and column \(j\).

16. \(\det B(i, j, r) = 1\) if \(i \neq j\); hence \(B(i, j, r)\) is non-singular for \(i \neq j\). In fact, 
\(B(i, j, r)^{-1} = B(i, j, -r)\).

17. For an \(m \times n\) matrix \(M\), let \(B(i, j, r)\) be the \(m \times m\) identity matrix with the element in row \(i\) and column \(j\) replaced with \(r\) \((i \neq j)\). Then left multiplication of \(M\) by \(B(i, j, r)\) replaces row \(i\) of \(M\) with row \(i\) plus \(r\) times row \(j\).

18. For an \(m \times n\) matrix \(M\), let \(C(i, j)\) be the \(m \times m\) identity matrix with rows \(i\) and \(j\) interchanged. Then left multiplication of \(M\) by \(C(i, j)\) interchanges rows \(i\) and \(j\) of \(M\). \(\det C(i, j) = -1\) for \(i \neq j\), and \(C(i, j)^{-1} = C(j, i)\).

19. Maximize \(c\) subject to
\[
\begin{bmatrix}
1 & 2 & 1 & 0 \\
1 & 1 & 0 & 1 \\
30 & 50 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = \begin{bmatrix}
24 \\
20 \\
i \\
\end{bmatrix}
\]
and \(x_1, x_2, x_3, x_4 \geq 0\) (\(x_5\) and \(x_6\) are slack variables).
I. Start with $x_1 = x_2 = 0$, $ε = 0$, $x_3 = 24$, $x_4 = 20$.

II. Pivot on row 1 column 2, obtaining

$$
\begin{bmatrix}
\frac{1}{2} & 1 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & -\frac{1}{2} & 1 \\
5 & 0 & -25 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= 
\begin{bmatrix}
12 \\
8 \\
\varepsilon - 600
\end{bmatrix}.
$$

I. New values: $x_1 = x_3 = 0$, $ε = 600$, $x_2 = 12$, $x_4 = 8$.

II. Pivot on row 2 column 1, obtaining

$$
\begin{bmatrix}
0 & 1 & 1 & -1 \\
1 & 0 & -1 & 2 \\
0 & 0 & -20 & -10
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
16 \\
\varepsilon - 680
\end{bmatrix}.
$$

I. New values: $x_1 = x_4 = 0$, $ε = 680$, $x_1 = 16$, $x_2 = 4$. Since

$$\varepsilon = 680 = 20x_1 = 40x_4 \quad \text{and} \quad x_1, x_4 \geq 0,$$

then $ε = 680$ is maximum.

Therefore, the maximum profit per day is $6.80, with 16 pounds of sugar cookies, and 4 pounds of gingerbread cookies.