1. The Problem and the Model

Let us consider the problem of calculating how long a traffic light should remain amber before turning red. Essentially, the amber cycle exists to allow vehicles in the intersection, or those too close to stop, to clear the intersection. Thus the light should remain amber long enough so that all drivers who cannot stop have a chance to pass through the intersection on the amber. A driver approaching an intersection should never be in the dilemma of being too close to stop safely and yet too far away to pass through the intersection before the red phase starts.

A "rule of thumb" exists for this calculation. Allow 1 s of amber for each 10 mi/h legal approach speed. Let us see if a theoretical calculation confirms this rule of thumb.

A driver approaching an intersection who gets an amber signal has a decision to make: whether to stop or to pass through the intersection. If he is traveling at the legal speed (or below it), he must have an adequate distance in which to stop. If he should decide to stop. If he should decide to pass through the intersection, he must have adequate time to pass completely through the intersection, including a time interval in which to decide to stop (reaction time) and the time it would take to drive the minimum distance needed to stop. Hopefully, drivers who see the amber soon enough will use the braking distance to stop their vehicles.

Thus, the amber phase should be long enough to include a driver's reaction time, plus the time it would take him to drive through the intersection, plus the time it would take him to drive the distance which would be required to brake his vehicle to a halt (the braking distance). Drivers having this much time would be able to stop safely within the braking distance.

If the legal speed is $v_0$, the width of the intersection is $l$, the distance to the intersection is $(L + L)/v_0$. (Note that the rear end of the vehicle must clear the intersection, thus the effective length of the intersection is $L + L$.)

Now, let us compute the braking distance. We note that the actual braking and stopping process is quite complicated, with drivers decelerating by first releasing the pressure on the accelerator, then stepping on the brake pedal with varying degrees of hardness (perhaps pumping it) until the vehicle stops. We shall bypass most of these processes; they are indeed difficult to model exactly.

We shall assume that the effect of braking the vehicle can be modeled by the introduction of a resisting frictional force upon the application of the brakes. Suppose $W$ is the weight of the vehicle and $f$ is the coefficient of friction. Then by definition, the braking force on the vehicle is $fW$, opposite the direction of motion (see Figure 10.1). The distance traveled in stopping can be found by solving the differential equation of motion in a straight line subject to a constant force $-fW$.

$$\frac{W}{g} \frac{dx}{dt^2} = -fW,$$

where $g$ is the gravitational acceleration.

The correct conditions to impose on $x$ are that at $t = 0$, $x = 0$, and $dx/dt = v_0$. The braking distance, then, is the distance traveled when $dx/dt = 0$.

2. The Solution

Integrating (1) subject to $dx/dt(0) = v_0$ gives

$$\frac{dx}{dt} = -gt + v_0.$$

* Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, NY 12181.