SECTION 1: CLASS NOTES

This section contains the class notes for the course.

1.1. Notes for Chapter 1: Introduction

1.1.1. Introduction (1.1-1.3 of the Text)

1. What is logic?

1.1 Arguments

(1) Some examples of arguments

Mary will marry John only if John loves her.
John loves Mary.

Therefore, Mary will marry John.

All human beings are mortal.
Socrates is a human being.

Therefore, Socrates is mortal.

If you can win the game, I would be the uncle of monkey.
......

(Therefore, you will not win the game.)

I will die if I am killed.
I am not killed.

Therefore, I will not die.

All the students in the room are logic students.
Some logic students are really boring.

Some students in the room are boring.

Swan a is white.
Swan b is white.
......
Swan n is white.

Therefore, all swans are white.

(2) Components of arguments

Definition: An argument is a group of statements, one or more of which (the premises) are claimed to provide support for, or reasons to believe, one of the others (the conclusion).
The structure of an argument:

Premise 1
Premise 2

\[ \text{support} \]

Conclusion

Premises provide some grounds (not necessarily guarantee) for the truths of the conclusion. There is an inferential relationship between premises and conclusion.

(3) Deductive vs. Inductive Arguments

1.2 Definition:

Logic is a subject (an art?) of the study of the methods and principles used to distinguish good / cogent from bad / fallacious argument.

2. How to evaluate (deductive) arguments: validity and soundness

2.1 Two basic criteria of evaluation

- Validity--the inferential relationship between Ps and C: Whether Ps support C and to what extent?
- Soundness—the status of premises: whether Ps are true or acceptable?

A good argument: (a) All Ps are acceptable (true) and (b) Ps support C to the extent that if all Ps are true, then it is impossible for C to be false.

2.2 Validity

Definitions:
- An Argument is valid if and only if it is \textit{logically impossible} for the conclusion to be false if all the premises to be true.
- An argument is valid iff the truths of the premises guarantee the truth of the conclusion.

A few feature of validity:
- Truth-preserving: from the truth of the premises to the truth of the conclusion.

- Hypothetical situation: Suppose / assume that all the premises are true, not that all the premises are actually true. For example, the following argument is valid although all the premises are actually false:

  All cats are sea creatures. (False)
  All sea creatures are cold-blooded killers. (False)

  \[ \text{All cats are cold-blooded killers. (False)} \]
All or nothing issue: validity has no degree.
Validation of an argument is determined by the form of the argument only (the inferential relation between the conclusion and the premises). Validity of an argument has nothing to do with the contents, and therefore the actual truth values, of the premises and the conclusion.

Examples:

<table>
<thead>
<tr>
<th>Argument Form</th>
<th>Arguments in English</th>
</tr>
</thead>
<tbody>
<tr>
<td>All S are M</td>
<td>All cats are sea creatures. (F)</td>
</tr>
<tr>
<td>All M are P</td>
<td>All sea creatures are mammals. (F)</td>
</tr>
<tr>
<td>All S are P</td>
<td>All cats are mammals. (T)</td>
</tr>
<tr>
<td></td>
<td>All cats are animals. (T)</td>
</tr>
</tbody>
</table>

**Valid Form**

<table>
<thead>
<tr>
<th>Valid Form</th>
<th>Arguments in English</th>
</tr>
</thead>
<tbody>
<tr>
<td>All S are M</td>
<td>All cats are sea creatures. (F)</td>
</tr>
<tr>
<td>All M are P</td>
<td>All sea creatures are mammals. (F)</td>
</tr>
<tr>
<td>All S are P</td>
<td>All cats are mammals. (T)</td>
</tr>
<tr>
<td></td>
<td>All cats are animals. (T)</td>
</tr>
</tbody>
</table>

**Invalid Form**

<table>
<thead>
<tr>
<th>Invalid Form</th>
<th>Arguments in English</th>
</tr>
</thead>
<tbody>
<tr>
<td>All S are M</td>
<td>All cats are mammals. (T)</td>
</tr>
<tr>
<td>All P are M</td>
<td>All dogs are mammals. (T)</td>
</tr>
<tr>
<td>All S are P</td>
<td>All cats are animals. (T)</td>
</tr>
<tr>
<td></td>
<td>All mammals are animals. (T)</td>
</tr>
</tbody>
</table>

### 2.3 Soundness

Definition: An argument is sound iff it is valid and all its premises are true.

\[
\text{Soundness} = \text{validity} + \text{truth of Ps.}
\]

### 3. How to determine whether an argument is valid?

Two steps of evaluation of validity:

Step I—**Symbolization** / translation: symbolize arguments in English into logical notation.

Example:
**Argument in English**

<table>
<thead>
<tr>
<th>Argument in English</th>
<th>Argument in Logical notions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mary will marry John <em>only if</em> John loves her. John loves Mary.</td>
<td>Marry (Mary, John) → Love (John, Mary) Love (John Mary)</td>
</tr>
<tr>
<td>Therefore, Mary will marry John.</td>
<td>Marry (Mary, John)</td>
</tr>
<tr>
<td>M → L</td>
<td>L</td>
</tr>
<tr>
<td>L</td>
<td>M</td>
</tr>
</tbody>
</table>

All the students in the room are logic students. Some logic students are really boring. Some students in the room are boring.

\[ \forall x [(S(x) \land I(x)) \rightarrow L(x)] \]

\[ \exists x [L(x) \land B(x)] \]

\[ \exists x [(S(x) \land I(x)) \land B(x)] \]

Step II—**Formal proof**: using some formal methods to determine the validity of the argument in logical notion.

- **Formal methods**
  - truth-tree method
  - truth-table method
  - natural derivation

4. **The function of artificial / formal language (the language of first-order logic or FOL) in symbolic sciences** (see 1.2)
   - in philosophy
   - in computer science
   - in math.
   - In AI (thinking = mental presentation + computation)
   - In linguistics

5. **Logic in rational inquiry** (see the text 1.1)

6. **Play with Tarski’s World**
1.2. Notes for Chapter 2: Atomic Sentences

1.2.1. The Basic Structure of Atomic Sentences
(2.1, 2.2, 2.3, and 2.5 of the Text)

1. Comparison between simple English sentences and atomic sentences

<table>
<thead>
<tr>
<th>Simple English Sentences (subject-predicate sentences)</th>
<th>Atomic sentences (FOL)</th>
</tr>
</thead>
<tbody>
<tr>
<td>John is a freshman</td>
<td>Freshman (John)</td>
</tr>
<tr>
<td>John swims.</td>
<td>Swim (John)</td>
</tr>
<tr>
<td>John loves Jenny.</td>
<td>Love (John, Jenny)</td>
</tr>
<tr>
<td>John prefers Jenny to Amy.</td>
<td>Prefer (John, Jenny, Amy)</td>
</tr>
<tr>
<td>John’s mother loves Jenny.</td>
<td>Love (mother (John), Jenny)</td>
</tr>
<tr>
<td>The father of Jenny is angry.</td>
<td>Angry (father (Jenny))</td>
</tr>
<tr>
<td>John is the brother of Jenny.</td>
<td>John = brother (Jenny)  [relational identity]</td>
</tr>
</tbody>
</table>

2. Names

Definition: Names are individual constants that refer to some fixed individual objects or other.

(1) The rule of naming (p. 10)
- No empty name.
- No multiple references (do not use one name to refer to different objects).
- Multiple names: you can name one object by different names.

(2) General terms / names: using a predicate, instead of a constant, to represent a general term. For example,
John is a student   Student (John) [correct]  John = student [wrong!!!]

3. Predicates

Definition: Predicates are symbols used to denote some property of objects or some relationship between objects.

(1) Arity of predicates
- Unary predicates--property
- Binary predicates
- Ternary predicates

(2) The predicates used in Tarski’s World: see p. 11.

(3) Two rules of predicates: see p.12.
4. Functions

Definition: A function is an individual constant determined by another constant.

(1) Comparison with names:
- Both refer to some fixed individual objects.
- Function is more complex than names: the reference of a function is determined by the relation to another name.

(2) Comparison with predicates:
- A predicate represents a property or a relation while a function denotes a fixed individual object.
- The same input but different output:

<table>
<thead>
<tr>
<th>Input</th>
<th>Program</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>object a</td>
<td>Function (the mother of a)</td>
<td>object b (the mother of a)</td>
</tr>
<tr>
<td>object a</td>
<td>Predicate (a is a mother)</td>
<td>a sentence (describe a state of affair: “a is a mother” or “a has a property of motherhood)</td>
</tr>
</tbody>
</table>

(2) Arity of function:

<table>
<thead>
<tr>
<th>English</th>
<th>Translation</th>
<th>FOL</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 plus 2</td>
<td>sum (2, 2) or + (2, 2)</td>
<td>sum (x, y)</td>
</tr>
<tr>
<td>the daughter of John and Amy</td>
<td>daughter (John, Amy)</td>
<td>daughter (x, y)</td>
</tr>
<tr>
<td>between block a and block b</td>
<td>between (a, b)</td>
<td>between (x, y)</td>
</tr>
</tbody>
</table>

5. Simple / Atomic sentences

Definition: a sentence consists of some names connected by a single predicate only.

(1) Three rules about simple sentences: see p. 13.
(2) An simple /atomic sentence expresses a claim that is either true or false (having a determinant truth value).

6. Elements of atomic sentences and translation

(1) The following is an example of a translation manual of sample sentences in 1.
## Translation Manual

<table>
<thead>
<tr>
<th>Components</th>
<th>English</th>
<th>FOL</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Names</strong></td>
<td>John</td>
<td>John</td>
</tr>
<tr>
<td></td>
<td>Jenny</td>
<td>Jenny</td>
</tr>
<tr>
<td></td>
<td>Amy</td>
<td>Amy</td>
</tr>
<tr>
<td><strong>Predicates</strong></td>
<td>x is a freshman</td>
<td>Freshman (x)</td>
</tr>
<tr>
<td></td>
<td>x is angry</td>
<td>Angry (x)</td>
</tr>
<tr>
<td></td>
<td>x swims</td>
<td>Swim (x)</td>
</tr>
<tr>
<td></td>
<td>x loves y</td>
<td>Love (x, y)</td>
</tr>
<tr>
<td></td>
<td>x is y</td>
<td>x = y</td>
</tr>
<tr>
<td></td>
<td>x prefer y to z</td>
<td>Prefer (x, y, z)</td>
</tr>
<tr>
<td><strong>Functions</strong></td>
<td>x’s mother</td>
<td>mother (x)</td>
</tr>
<tr>
<td></td>
<td>the father of x</td>
<td>father (x)</td>
</tr>
<tr>
<td></td>
<td>the brother of x</td>
<td>brother (x)</td>
</tr>
</tbody>
</table>

(2) The procedure of translation:

Simple English sentences (the sentences on the left of hand side of 1) → Translation manual → atomic sentences in FOL (the sentences on the left hand side of 1)

### 7. A classification of terms

Definition: A term is a noun / noun phrase or expression used to refer to an individual objects (either fixed or unfixed) object.

Terms

- **Fixed terms**
  - Simple terms: names / constants such as John.
  - Complex terms (name-like terms): functions (function symbols + terms) such as the father of John

- **Unfixed terms**: variables such as x.
8. Class exercises

**Problem 9** (p. 19). Translation between the relational language and the functional language.
First of all, make a list of all available symbols in each language:

<table>
<thead>
<tr>
<th></th>
<th>the relational language</th>
<th>the functional language</th>
</tr>
</thead>
<tbody>
<tr>
<td>Names</td>
<td>Claire, Melanie, Jon</td>
<td>Claire, Melanie, Jon</td>
</tr>
<tr>
<td>Predicates</td>
<td>TallerThan(x, y)</td>
<td>TallerThan(x, y)</td>
</tr>
<tr>
<td></td>
<td>FatherOf (x, y): x is the father of y.</td>
<td>x = y</td>
</tr>
<tr>
<td>Functions</td>
<td>father (x): the father of x</td>
<td></td>
</tr>
</tbody>
</table>

Second, translate from the relational language to the functional language:
1. FatherOf (Jon, Claire)  Jon = father (Claire)

2. FatherOf (Jon, Melanie) Jon = father (Melanie)

3. TallerThan (Melanie, Claire) TallerThan (Melanie, Claire)

Third, only sentence 1 can be translated into atomic sentence of the relational language.

1. father (Melanie) = Jon FatherOf (Jon, Melanie)

2. “father (Melanie) = father (Claire)” means “Melanie’s father is Claire’s father” or “Melanie and Claire have the same father.”

3. “TallerThan (father (Claire), father (Jon))” means “Claire’s father is taller than Jon’s father.”

**Problem 10** (p. 20)

First of all, make a translation manual as follows:

<table>
<thead>
<tr>
<th></th>
<th>English</th>
<th>FOL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Names</td>
<td>Carl, Sam, Mary</td>
<td>the same</td>
</tr>
<tr>
<td>Predicates</td>
<td>x is the same as y</td>
<td>x = y</td>
</tr>
<tr>
<td></td>
<td>x is greater than y</td>
<td>x &gt; y</td>
</tr>
<tr>
<td>Functions</td>
<td>the height of x</td>
<td>height (x)</td>
</tr>
</tbody>
</table>

Then, translate sentences into logical notations, for example:
Carl is taller than Sam.  \[ \text{height (Carl)} > \text{height (Sam)} \]

Sam and Mary are the same height.  \[ \text{height (Sam)} = \text{height (Mary)} \]

Mary is shorter than Carl.  \[ \text{height (Carl)} > \text{height (Mary)} \]

1.2.2. Translating Simple English Sentences into Logical Notation (2.7 of the Text)

There are two kinds of translation problems:
- Based on predefined language or translation manual.
- Create your own language or translation manual.

1. Translation based on a given translation manual
   Problem 15 (p. 23)
   1. Owned (Claire, Folly, 2:00)
   2. Gave (Claire, Silly, Max, 2:05)
   3. Student (Max)
   4. Erased (Claire, Folly, 2:00)
   5. Owned (Max, Folly, 3:05)
   6. \[ 2:00 < 2:05 \]

   Problem 16 (p. 23)
   1. Max owned Silly at 2:00 pm.
   2. Max erased Silly at 2:00 pm.
   3. Max gave Silly to Claire at 2:00 pm.
   2:00 pm is earlier than 2:00 pm.

2. Translation by making a translation manual
   A few tips of making up a manual:
   - If possible, adopt a predicate with a bigger arity.

   \[ \text{x prefers Jenny to y.} \]
   John prefers Jenny to Mary \[ \text{x prefers y to z.} \]

   - Name could be used to refer to abstract objects (time, place, Sunday)
   - Use as less logical symbols as possible.
Problem 17 (p. 24)
(1) One suggested translation manual.

<table>
<thead>
<tr>
<th><strong>English</strong></th>
<th><strong>FOL</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Names</strong></td>
<td><strong>Claire, John, Jon, Nancy, Max, Mary Ellen France, Spain, Portugal, AIDS, influenza Company, Misery</strong></td>
</tr>
<tr>
<td><strong>Predicates</strong></td>
<td><strong>x is between y and z in size.</strong>&lt;br&gt;x is less contagious than y.&lt;br&gt;x loves y&lt;br&gt;x shook y&lt;br&gt;x is younger than y.**&lt;br&gt;<strong>BetweenInSize (x, y, z)</strong>&lt;br&gt;<strong>LessContagious (x, y)</strong>&lt;br&gt;<strong>Love (x, y)</strong>&lt;br&gt;<strong>Shook (x, y)</strong>&lt;br&gt;<strong>YoungerThan (x, y)</strong></td>
</tr>
<tr>
<td><strong>Functions</strong></td>
<td><strong>the father of x</strong>&lt;br&gt;<strong>the hand of x</strong>&lt;br&gt;<strong>the eldest child of x and y</strong>&lt;br&gt;<strong>father (x)</strong>&lt;br&gt;<strong>hand (x)</strong>&lt;br&gt;<strong>eldestChild (x, y)</strong></td>
</tr>
</tbody>
</table>

(2) Translation based on the above manual:
1. LessContagious (AIDS, Influenza)
2. Between (Spain, France, Portugal)
3. Loves (Misery, Company)
4. Shook (Max, hand (father (Claire)))
5. YoungerThan (eldest (John, Nancy), eldest (Jon, Mary Ellen))

1.2.3. Methods of Proof (2.8 of the Text)

1. **Logical Consequence**

The conclusion is a logical consequence of its premises iff the argument is valid.
A statement C is a logical consequence of a set of statements \( \{P\} \) iff \( \{P\} \) are true, then C must be true.

\( \{P_1, P_2, ..., P_n\} \models C \)

For example,

\( \{\text{if A, then B}; A\} \models B \quad \text{If A, then B} \)

\( A \)

\( B \)
2. A Proof

2.1 Definition

A proof is a step-by-step demonstration that a given conclusion (say C) follows from some premises \{P1, P2, and P3\} in any circumstance.

If you can give a proof that C follows from \{P1, P2, P3\}, then C is the logical consequence of \{P1, P2, P3\}. Accordingly, the argument P1, P2, P3 / C is valid.

For example,

(1) If I study hard enough for a course, then I will pass it.
(2) I study hard enough for symbolic logic course.
(3) I will graduate this fall if I pass symbolic logic course.
(4) I will graduate this fall.

Rule used: \[ \text{If } A, \text{ then } B \]

A

B

“Modus Ponens” or “Conditional Elimination”

2.2 Informal vs. Formal Proofs

How can we prove that \{Cube (a), a = b\} ⊨ Cube (b)?

Informal proof:

Suppose that Cube (a) and a = b. According to the principle of indiscernibility of identicals, since a = b, we can replace a in Cube (a) by b. We come up with Cube (b), as desired.

- Used by mathematicians;
- Stated in natural language;
- More free-wheeling style;
- Not always include every step.

Formal proof:

\[
\begin{align*}
1. \text{ Cube (a)} \\
2. \text{ a = b} \\
3. \text{ Cube (b)} \quad \text{Ind Id: 1, 2}
\end{align*}
\]

- Used by logicians;
- Stated in FOL;
- Follow stylized method of presentation, such as “Fitch-style” system used by our book.
3. Informal Proofs Involving Identity

Any proof depends on a few derivation rules / proof rules which are either self-evident or proved logical rules. The essential property of these rules is truth preserving. That is, these rules can pass the truth of one statement to another. We will introduce many derivation rules in the due procedure.

For atomic sentences, there are four such derivation rules as follows:

(1) The principle of Reflexivity of Identity (Refl=): \( \phi \models a = a \) (“a” and “a” have the same referent)

(2) The principle of Indiscernibility of Identicals (Ind Id): Two identicals always have the same property. That is, if \( a = b \), then \( P(a) = P(b) \). Or in symbols, \( a = b \models P(a) = P(b) \).

For example, suppose that Bill is the father of John and Bill is the president of Juniata College. Then the father of John is the president of Juniata College.

Be careful here: Only the RIGHT hand symbol of an identity can be used for substitution!! For example, from \( a = b \) and \( P(a) \), we can directly derive \( P(b) \). But from \( a = b \) and \( P(b) \), we CANNOT derive \( P(a) \) directly. For the second proof, we have to make a detour with the help of Refl=: First, suppose that \( a = b \). We know that \( a = a \) by Refl=. Now replace the first use of \( a \) in \( a = a \) by \( b \) in \( a = b \) by Refl=. We come up with \( b = a \), as desired. Second, from \( b = a \) and \( P(b) \), we have \( P(a) \) by Ind Id.

(3) The principle of the Symmetry of Identity (Sym Id): \( a = b \models b = a \)

(4) The principle of Transitivity (Tran): \( \{a = b, \ b = c\} \models a = c \)

Other similar principles of transitivity involving relationship between objects, such as “greater than” or “less than” relationship.

\[ \{\text{Larger} (a, b), \text{Larger} (b, c) \} \models \text{Larger} (a, c) \]
\[ \{\text{LessThan} (a, b), \text{LessThan} (b, c) \} \models \text{LessThan} (a, c) \]

4. Class exercises

Problem 18 (p. 30)
Proof: Suppose that \( a = b \) and \( b = c \) (two given premises). According to Ind Id, we can replace \( b \) in \( a = b \) by \( c \). We come up with \( a = c \), as desired.

Problem 19 (p. 30)
Proof: Suppose that LeftOf(a, b). By the meaning of LeftOf and RightOf in the language of Tarski’s World, we have RightOf (b, a).
1. Proof: Suppose that LeftOf (a, b) and \( b = c \). Using Ind Id, we have LeftOf(a, c) by replacing \( b \) in RightOf(a, b) by \( c \) in \( b = c \). By the meaning of LeftOf and RightOf in Language of TW, we get RightOf(c, a), as desired.
2. The alleged conclusion is not the logical consequence of the premises. So no proof.
3. Proof: Suppose that BackOf(a, b) and FrontOf(a, c). By the meaning of BackOf and FrontOf in the language of TW, BackOf(a, b) is equivalent to FrontOf(b, a). We know that \( b \) is in front of \( a \) and \( a \) is in front of \( c \). So it follows that \( b \) is in front of \( c \) (transitivity). That is, Front Of (b, c), as desired.
4. The alleged conclusion is not a logical consequence of the premises. No logical proof.
Problem 20 (p. 30)
We assume that only one person can own a disk at any given time.
1. (3) does follow from (1)m and (2).
2. No! For example, let Folly belong to Claire at 2 pm, and let Silly belong to Mary at 2 pm. So, under this situation, (1) is true, (3) is True, but (2) is false.
3. No! For example, let Folly and Silly belong to Max at 2 pm. Then (2) is true and (3) is true. But (1) is false in this case.

1.2.4. Formal Proofs (2.9 of the Text)

1. The structure of a formal proof in “Fitch-Style” system

\[
\begin{align*}
1. & \text{ P1 } \\
2. & \text{ P2 } \text{ premises} \\
3. & \text{ P3 } \\
\vdots & \text{ intermediate conclusions} \\
\vdots & \\
\vdots & \\
\#n & \text{ C final conclusion}
\end{align*}
\]

2. Derivation Rules

Rule 1: Reflexivity of Identity (Refl=)

\[
\begin{align*}
\Diamond & \text{ a = a no justification needed}
\end{align*}
\]

Rule 2: Indiscernibility of Identicals (Ind Id)

\[
\begin{align*}
\#m & \text{ P(a) } \\
\vdots & \\
\vdots & \\
\#n & \text{ a = b } \\
\vdots & \\
\vdots & \\
\Diamond & \text{ P(b) Ind Id: #m, #n } \\
\end{align*}
\]

the # of Identity

Examples:

\[
\begin{align*}
1. & \text{ Cube (a) } \\
2. & \text{ a = b } \\
3. & \text{ Cube (b) Ind Id: 1, 2 } \\
\end{align*}
\]

\[
\begin{align*}
1. & \text{ a = b } \\
2. & \text{ a = a Refl=} \\
3. & \text{ b = a Ind Id: 2, 1 } \\
\end{align*}
\]

\[
\begin{align*}
1. & \text{ a = b } \\
2. & \text{ b = c } \\
3. & \text{ a = c Ind Id: 1, 2 } \\
\end{align*}
\]
Be careful here:
(1) You can substitute *some* or *all* of the occurrences of one name.

\[
\begin{align*}
&: \\
&\text{\#m } a = b \\
&: \\
&\text{\#n } P(a, a, c) \quad \text{Ind Id: \#n, \#m} \\
&\quad P(b, a, c) \\
&\quad P(a, b, c) \\
&\quad P(b, b, c) \\
&:
\end{align*}
\]

(2) Only a name on the RIGHT hand side of an identity can be used for substitution.

\[
\begin{align*}
&: \\
&a = b \\
&:\quad P(b) \quad \text{WRONG!!!} \\
&: \\
&P(a) \\
&:
\end{align*}
\]

\[
\begin{align*}
&: \\
&\text{\#m } a = b \\
&: \\
&\text{\#n } P(b) \\
&: \\
&\text{\#l } a = a \quad \text{Ref} = \\
&\text{\#h } b = a \quad \text{Ind Id: \#l, \#m} \\
&\quad P(a) \quad \text{Ind Id: \#n, \#h} \\
&:
\end{align*}
\]

Rule 3: Reiteration (Reit)

\[
\begin{align*}
&: \\
&P \\
&:\quad \Diamond P \quad \text{Reit}
\end{align*}
\]
3. Class exercises

(1) Prove \{\text{Like}(b, b), a = b\} \vdash \text{Like}(b, a)

1. \text{Like}(b, b)
2. a = b
3. a = a Refl=
4. b = a Ind Id: 3,2
5. \text{Like}(b, a) Ind Id: 1, 4

(2) Problem 21: Prove \{a = b, b = c\} \vdash a = c

1. a = b
2. b = c
3. a = c Ind Id: 1, 2

(3) Problem 22: Prove \{\text{Like}(a, b), b = c, c = d\} \vdash \text{Like}(a, d)

1. \text{Like}(a, b)       1. \text{Like}(a, b)
2. b = c         2. b = c
3. c = d         3. c = d
4. b = d   Ind Id: 2, 3    4. Like(a, c)  Ind Id: 1, 2
5. Like(a, d)  Ind Id: 1, 4     5. Like(a, d)  Ind Id: 4, 3

(4) Problem 23: Prove Between(a, d, b), a = c, e = b\} \vdash \text{Between}(c, d, e)

1. Between(a, d, b)
2. a = c
3. e = b
4. Between(c, d, b) Ind Id: 1, 2
5. Between(c, d, e) Ind Id: 4, 3 WRONG!!!

(5) Problem 24
First, set up a rule: Transitivity of Smaller (Trans. Sm)

\begin{align*}
\#m & \quad \text{Smaller}(a, b) \\
\vdots & \\
\#n & \quad \text{Smaller}(b, c) \\
\vdots & \\
& \quad \text{Smaller}(a, c) \quad \text{Trans. Sm:} \ #m, \ #n
\end{align*}

Then, prove: \{\text{Smaller}(a, b), \text{Smaller}(c, d), b = c\} \vdash \text{Smaller}(a, d)

1. Smaller(a, b)
2. Smaller(c, d)
3. b = c
4. Smaller(a, c) Ind Id: 1, 3
5. Smaller(a, d) Trans. Sm: 4, 2
1.3. Notes for Chapter 3: Conjunctions, Disjunctions, and Negations

1.3.1. Introduction to Conjunctions, Disjunctions, and Negations (3.1, 3.2, 3.3, and 3.4 of the Text)

1. Introduction
1.1 Moving from atomic sentences to compound sentences

Sentences

Simple sentences:  \{ one single predicate
                   no truth-functional connectives. \}

Compound sentences: \{ More than one predicate
                          Two or more simple sentences connected by some truth-functional connectives. \}

For example:

John is a student and Joe is a teacher.

John or Joe is a student = John is a student or Joe is a student.

If John is a student, then Joe is a teacher.

John is not a student.


Definition: a compound sentence is truth-functional iff the truth-value of the sentence is fully determined by the truth-value of its component simple sentences.

Then the connectives connected component sentences of a truth-functional sentence is truth-functional connectives. They are: conjunction, disjunction, negation, and conditional.

For example,

Truth-functional sentences:

<table>
<thead>
<tr>
<th>John is a student and Joe is a teacher.</th>
<th>(False)</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
<td>False</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>If John is a student, then Joe is a teacher.</th>
<th>(False)</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
<td>False</td>
</tr>
</tbody>
</table>
Non-truth-functional sentences:

John loves Kathy because he kisses her.  (True or False)  
True

I believe that Pat is on the mat (propositional attitude).  (True or False)  
True

2. Syntax of truth-functional connectives

<table>
<thead>
<tr>
<th>Conjunction</th>
<th>In English</th>
<th>In FOL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>and, but, however, although,</td>
<td>( \wedge ) / &amp;</td>
</tr>
<tr>
<td></td>
<td>nevertheless, moreover, in additions, ,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Disjunction</td>
<td>or, either...or..., at least one of two..., unless</td>
</tr>
<tr>
<td></td>
<td>Negation</td>
<td>not, be hardly, unhappy, impossible, incomplete</td>
</tr>
</tbody>
</table>

Attention: two senses of disjunction

Disjunction

- In exclusive sense: exactly one alternative (at least one and at most one alternative)
- In inclusive sense: at least one alternative (and could be both)

For example:

- Waitress: “You can have ice cream or a cake as desert” (but not both).
- Alice and Katy’s father: “John, you can marry either Alice or Kathy” (but not both).
- Professor: “Joe, you can take either ethics or human nature course to fulfill your philosophy requirement” (sure you can take both if you like).

In FOL, we define “OR” in inclusive sense only.

3. Semantics of truth-functional connectives

3.1 Truth-table definitions: Suppose that P and Q here represent any sentence (either simple or compound sentences). We can define the truth-value of a compound sentence consisting of P and Q as follows:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P ( \wedge ) Q</th>
<th>P ( \lor ) Q</th>
<th>( \neg ) P</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>
3.2 How to determine the truth value of a sentence?

(1) If the truth-values of component sentences of a compound sentence are given: From inside to outside! (when you determine the truth value of a sentence based on a Tarski’s World).

\[
\begin{array}{c}
(P \land Q) \land (Q \lor R) \\
T \ F \ F \ F \ T \ T \\
\neg (P \land Q) \land (R \lor P) \\
T \ F \ F \ T \ T \ T \ T \ F
\end{array}
\]

(2) If the truth value of a compound sentence is given: from outside to inside! (when you play the game to see your truth commitments)

\[
\begin{array}{c}
(P \land Q) \land (Q \lor R), \ therefore, \ P \ is \ true, \ Q \ is \ true, \ and \ R \ could \ be \ true \ or \ false. \\
T \ T \ T \\
\neg (P \land Q) \land (R \lor P) \\
F
\end{array}
\]

In this case, there are four possible truth values:
- R is false and P is false, and Q could be true or false. OR
- P is true and Q is true, and R could be either true or false.

4. Correct use of parentheses

\[
\begin{array}{c}
P \land Q \lor R \ \text{(WRONG!)} \\
(P \land Q) \lor R \\
P \land (Q \lor R)
\end{array}
\]

Conventions:
- “\(\neg\)” always apply to the smallest unit right after it:

\[
\begin{array}{c}
\neg R \lor (P \lor Q) \\
\neg (R \lor (P \lor Q)) \\
\neg (R \lor P) \lor Q
\end{array}
\]

- R \lor P \lor Q \quad \text{Okay!!}
- P \land Q \land R \quad \text{Okay!!}
1.3.2. Logical Equivalency (3.5 of the Text)

1. Definition:

   Two sentences are logically equivalent iff they have same truth value in exactly the same circumstances (under any possible interpretation / in any possible world).

Illustrations:

1.1 In the language of the Tarski’s World (with fixed interpretation)

   \(a\) is to one side or other of cube \(b\), but is in front of dodecahedron \(c\).

Suppose that you have two different translations of the above English sentence as follows:

(a) \([\text{Cube}(b) \land (\text{LeftOf}(a, b) \lor \text{RightOf}(a, b)) ] \land [\text{Dodec}(c) \land \text{FrontOf}(a, c)]\)

(b) \([ (\text{Cube}(b) \land \text{LeftOf}(a, b)) \lor (\text{Cube}(b) \land \text{RightOf}(a, b)) ] \land [\text{Dodec}(c) \land \text{FrontOf}(a, c)]\)

Are sentence (a) and (b) logically equivalent? To determine this, you need to see whether they always have the same truth value in any Tarski’s world. If they always have the same truth value in any Tarski’s world, then they are logically equivalent (Question: How can you do this??). If they do not have the same truth value in one Tarski’s world, then they are not logically equivalent (an counterexample).

1.2 In any formal language

   \(\neg (P(a) \land Q(a))\) \hspace{1cm} \(\neg P(a) \land \neg Q(a)\)

Are the above sentences in FOL logically equivalent? To find out, let us give a possible interpretation to the predicates and names under considerations.

Suppose:

\[
\begin{align*}
    P(x): \, & x \text{ is a student.} \\
    Q(x): \, & x \text{ is a female.} \\
    a: \, & \text{Sean}
\end{align*}
\]

Under this interpretation, sentence (a) means that it is not the case that Sean is a female student or Sean is not a female student (but Sean may be a student). Sentence (b) says that Sean is not a female, and Sean is not a student either. Suppose further that Sean is a male student in a possible world. Then under the above interpretation and in the above circumstance, sentence (a) is true but sentence (b) is false.

In conclusion, sentences (a) and (b) are not logically equivalent (since we have found one possible world in which they do not have the same truth value).

2. How to test logical equivalency?

   There are many different formal methods to test for logical equivalency. We will only introduce two of them at this stage.
2.1 Truth-table method
Make a truth table for both sentences to be tested. If they have the same truth values at all rows, then they are logically equivalent. Otherwise they are not.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>¬(P ∧ Q)</th>
<th>¬P ∨ ¬Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Conclusion: “¬(P ∧ Q)” and “¬P ∨ ¬Q” are logically equivalent. That is,

¬(P ∧ Q) ⇔ ¬P ∨ ¬Q

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>¬(P ∧ Q)</th>
<th>¬P ∧ ¬Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Conclusion: “¬(P ∧ Q)” and “¬P ∧ ¬Q” are not logically equivalent.

2.2 Rules of logical equivalency
Rules:
(1) Double negation: ¬¬P ⇔ P
(2) DeMorgan Rules:
¬(P ∧ Q) ⇔ ¬P ∨ ¬Q
¬(P ∨ Q) ⇔ ¬P ∧ ¬Q
• Negation normal form: “¬” only applies to atomic sentences.

(3) Idempotence: P ∧ P ⇔ P
P ∨ P ⇔ P

(4) Commutative rules:
P ∧ Q ∧ R ⇔ R ∧ P ∧ Q
P ∨ Q ∨ R ⇔ Q ∨ R ∨ P

(5) Association rules:
(P ∧ Q) ∧ R ⇔ P ∧ (Q ∧ R)
P ∨ (Q ∨ R) ⇔ (P ∨ Q) ∨ R

(6) Distribution rules:
P ∧ (Q ∨ R) ⇔ (P ∧ Q) ∨ (P ∧ R)
P ∨ (Q ∧ R) ⇔ (P ∨ Q) ∧ (P ∨ R)

• Disjunctive normal form: disjunction of conjunction of literals.
• Conjunctive normal form: conjunction of disjunction of literals.
Examples:

(1) Put $\neg [(A \land B) \land (C \lor \neg D)]$ into negation normal form.

$\neg [(A \land B) \land (C \lor \neg D)]$

$\iff \neg (A \land B) \lor \neg (C \lor \neg D)$

$\iff (\neg A \lor \neg B) \lor (\neg C \land \neg \neg D)$

$\iff (\neg A \lor \neg B) \lor (\neg C \land D)$

(2) Put $(A \lor B) \land C \land [(\neg (\neg B \land \neg A) \lor B]$ into negation normal form

$(A \lor B) \land C \land [(\neg (\neg B \land \neg A) \lor B]$

$\iff (A \lor B) \land C \land [(\neg \neg B \lor \neg \neg A) \lor B]$

$\iff (A \lor B) \land C \land [(B \lor A) \lor B]$

$\iff (A \lor B) \land C \land (B \lor A \lor B)$

$\iff (A \lor B) \land C \land (B \lor A)$

(3) Put $(A \lor B) \land (C \lor D)$ into disjunctive normal form.

$(A \lor B) \land (C \lor D)$

$\iff [(A \lor B) \land C] \lor [(A \lor B) \land D]$

$\iff [(A \land C) \lor (B \land C)] \lor [(A \land D) \lor (B \land D)]$

$\iff (A \land C) \lor (B \land C) \lor (A \land D) \lor (B \land D)$

(4) Prove that $\neg (A \lor B) \land \neg (B \lor C)$ and $\neg A \land \neg B \land \neg C$ are logically equivalent.
Prove:

$\neg (A \lor B) \land \neg (B \lor C)$

$\iff (\neg A \land \neg B) \land (\neg B \land \neg C)$

$\iff \neg A \land \neg B \land \neg B \land \neg C$

$\iff \neg A \land \neg B \land \neg C \quad \text{as desired.}$

(5) Prove that $(A \lor B) \land C \land (\neg (\neg B \land \neg A))$ and $(A \lor B) \land C$ are logically equivalent.
Prove:

$(A \lor B) \land C \land (\neg (\neg B \land \neg A))$

$\iff (A \lor B) \land C \land (\neg \neg B \land \neg \neg A)$

$\iff (A \lor B) \land C \land (B \lor A)$

$\iff (A \lor B) \land C \quad \text{as desired!}$
1.3.3. Translation (3.6 of the Text)

1. A standard of a correct translation

A (not the) logical symbolization of an English sentence is correct iff both are logically equivalent.

2. Tips for translations involving conjunctions, disjunctions and negations

(1) Identify the primary connective of the original English sentence to be translated.

Example:

Both d and c are cubes; moreover neither of them is small.

Primary

(Cube(d) ∧ Cube(c)) ∧ (¬ Small(d) ∧ ¬ Small(c))

(2) Paraphrase, if necessary, the sentences to be translated before translation (as long as they are both logically equivalent).

Example:
The original sentence: Neither e nor a is to the right of c and to the left of b

Rewrite the sentence as:

“Neither e is to the right of c and to the left of b nor a is to the right of c and to the left of b.”

“Neither e is to the right of c and to the left of b AND it is not the case that a is to the right of c and to the left of b.”

After the paraphrasing, translation is easy:

¬ [RightOf(e, c) ∧ LeftOf(e, b)] ∧ ¬ [RightOf(a, c) ∧ LeftOf(a, b)]

(3) A few common patterns of sentences

- Exclusive sense of OR, P or Q (but not both): (P ∨ Q) ∧ ¬ (P ∧ Q)
- Neither P not Q ⇔ It is not that either P or Q: ¬ P ∧ ¬ Q ⇔ ¬ (P ∨ Q)
- P unless Q: P ∨ Q
3. Class exercises

Problem 17 (p. 51)

A translation manual:

<table>
<thead>
<tr>
<th>English</th>
<th>FOL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Names</td>
<td>the same</td>
</tr>
<tr>
<td>Abe, AIDS, Al, Bill, Daisy, Dan, Dee, George, Influenza, Monday, Polonius, Stephen, Sunday,</td>
<td></td>
</tr>
<tr>
<td>Predicates</td>
<td>Admires(x, y)</td>
</tr>
<tr>
<td>x admires y</td>
<td>Borrower(x)</td>
</tr>
<tr>
<td>x is a borrower.</td>
<td>Fooled(x, y, z)</td>
</tr>
<tr>
<td>x fooled y on z</td>
<td>Jolly(x)</td>
</tr>
<tr>
<td>x is jolly</td>
<td>Lender(x)</td>
</tr>
<tr>
<td>x is a lender.</td>
<td>LessContagious(x, y)</td>
</tr>
<tr>
<td>x is less contagious than y</td>
<td>Lives(x, y)</td>
</tr>
<tr>
<td>x lives near y</td>
<td>Miller(x)</td>
</tr>
<tr>
<td>x is a miller.</td>
<td>MoreDeadly(x, y)</td>
</tr>
<tr>
<td>x is more deadly than y.</td>
<td>River(x)</td>
</tr>
<tr>
<td>x is a river.</td>
<td></td>
</tr>
<tr>
<td>Functions</td>
<td>eldestChild(x)</td>
</tr>
<tr>
<td>the eldest child of x</td>
<td></td>
</tr>
</tbody>
</table>

Translation:
1. LessContagious(AIDS, Influenza) ∧ MoreDeadly(AIDS, Influenza)
2. Fooled(Abe, Stephen, Sunday) ∧ ¬ Fooled(Abe, Stephen, Monday)
3. (Admire(Dan, Al) ∧ Admires(Dan, Bill)) ∨ (Admires(George, Al) ∧ Admires(George, Bill))
4. Jolly(Daisy) ∧ Miller(Daisy) ∧ River(Dee) ∧ Lives(Daisy, Dee)
5. ¬ (Borrower(eldestChild(Polonius)) ∨ Lender(eldestChild(Polonius)))
1.3.4. Formal Proofs (3.9 of the Text)

1. Derivation rules for conjunctions, disjunctions, and negations

1.1 Simple Rules (without subproofs)

(1) **Conjunction Elimination** (\(\land\) Elim)

\[
\begin{align*}
\text{\#m } & P \land R \land Q \\
\vdots & \\
\vdots & \\
\diamond & R \land \text{Elim: } \#m
\end{align*}
\]

Example (a): \(\{A \land B, C\} \vdash B \land C\)

1. \(A \land B\)
2. \(C\)
3. \(B \land \text{Elim: } 1\)
4. \(B \land C \land \text{Intro: } 3, 2\)

(2) **Conjunction Introduction** (\(\land\) Intro)

\[
\begin{align*}
\text{\#m } & P \\
\#n & R \\
\#l & Q \\
\vdots & \\
\diamond & P \land R \land Q \land \text{Intro: } \#m, \#n, \#l
\end{align*}
\]

Example (b): \(\{P \lor R, Q\} \vdash R \land (P \lor Q)\)

1. \(P \lor Q\)
2. \(R\)
3. \(R \land (P \lor Q) \land \text{Intro: } 2, 1\)

(3) **Disjunction Introduction** (\(\lor\) Intro)

\[
\begin{align*}
\text{\#m } & P \\
\vdots & \\
\vdots & \\
\diamond & P \lor R \lor Q \lor \text{Intro: } \#m
\end{align*}
\]

Example (c): \(\{P \land Q\} \vdash Q \lor R\)

1. \(P \land Q\)
2. \(Q \lor \text{Intro: } 1\)
3. \(Q \lor R \lor \text{Intro: } 2\)

(4) **Negation Elimination** (\(\neg\) Elim)

\[
\begin{align*}
\text{\#m } & \neg \neg P \\
\vdots & \\
\vdots & \\
\diamond & P \neg \text{Elim: } \#m
\end{align*}
\]

Example (d): \(\{
eg \neg (Q \land R), P\} \vdash (R \land P) \lor Q\)

1. \(\neg \neg (Q \land R)\)
2. \(P\)
3. \(Q \land R \neg \text{Elim: } 1\)
4. \(Q \land \text{Elim: } 3\)
5. \((R \land P) \lor Q \lor \text{Intro: } 4\)
1.2 Complex Rules (with subproofs)

(5) Disjunction Elimination

Two illustrations:
First, let us consider the so-called Disjunctive Dilemma: Protagoras vs. Euathlus
(a) Euathlus will either lose or win.
(b) If he loses the case, then he has to pay back my tuition (by the order of the court).
(c) If he wins the case, then he has to pay back my tuition also (by the terms of the contract).
(d) Either way, Euathlus has to pay back my tuition.

That is,
(a) Lose(Euathlus) ∨ Win (Euathlus)
(b) Lose (Euathlus) → PayBack(Euathlus, Tuition)
(c) Win (Euathlus) → PayBack (Euathlus, Tuition)
(d) PayBack (Euathlus, Tuition)

Secondly, suppose that we want to prove

\{ (Cube(c) ∧ Small (c)) ∨ (Tet(c) ∧ Small (c)) \} \vdash \text{Small (c)}

To prove it, let us break it into two cases, corresponding to the two disjuncts as follows:

1. Cube(c) ∧ Small (c)
2. Small (c)
3. Small (c) ∧ Elim: 1

There are only two alternatives. And in either case, we have Small (c). Then we have proved that Small (c) is a logical consequence of the premises.
Example (e): \{(A \wedge B) \vee (C \wedge D)\} \models B \vee D

Example (f): \{(A \wedge B) \vee C\} \models C \vee B

1. \((A \wedge B) \vee (C \wedge D)\)
   - 2. \(A \wedge B\)
   - 3. \(B \quad \wedge \text{Elim: 2}\)
   - 4. \(B \vee D \quad \vee \text{Intro: 3}\)
   - 5. \(C \wedge D\)
   - 6. \(D \quad \wedge \text{Elim: 5}\)
   - 7. \(B \vee D \quad \vee \text{Intro: 6}\)
   - 8. \(B \vee D \quad \vee \text{Elim: 1, 2-4, 5-7}\)

1. \((A \wedge B) \vee C\)
   - 2. \(C\)
   - 3. \(C \vee B \quad \vee \text{Intro: 2}\)
   - 4. \(A \wedge B\)
   - 5. \(B \quad \wedge \text{Elim: 4}\)
   - 6. \(C \vee B \quad \vee \text{Intro: 5}\)
   - 7. \(C \vee B \quad \vee \text{Elim: 1, 2-3, 4-6}\)

(6) **Negation Introduction** (¬ Intro)

\[\begin{array}{c}
\vdots \\
\ \ #m \quad P \\
\vdots \\
\ \ #n \quad Q \wedge \neg Q \quad \text{(a contradiction)} \\
\neg P \quad \neg \text{Intro: #m - #n}
\end{array}\]

Illustration: Method of proof by contradiction.
Suppose that we want to prove that \(\neg (b = c)\) is a logical consequence of \{\neg Tet(c), Tet(b)\}. To prove it, let us ASSUME (for the sake of argument) that \(b = c\), see what will follow from the assumption.

1. Tet(b)
2. \(b = c\)
3. Tet(c) \quad \text{Ind. Id: 1, 2}

But one premise says that \(\neg\text{Tet}(c)\). This contradicts the logical consequence of our assumption that \(b = c\). Therefore, our assumption that \(b = c\) cannot be true since it leads to a contradiction that \(\neg\text{Tet}(c) \wedge \text{Tet}(c)\). That means that \(b = c\) in a logical consequence of the original premises. That is what we want to prove.
Example (g): \( \{A\} \models \neg
\neg A \)

Example (h): \( \{P, \neg P\} \models Q \)

1. \( A \)
   2. \( \neg A \)
   3. \( A \land \neg A \land \text{Intro: 1, 2} \)
   4. \( \neg A \land \text{Intro: 2-3} \)

1. \( P \)
   2. \( \neg P \)
   3. \( \neg Q \land \text{Intro: 1, 2} \)
   4. \( P \land \neg P \land \text{Intro: 1, 2} \)
   5. \( \neg Q \neg \text{Intro: 3-4} \)
   6. \( Q \neg \text{Elim: 5} \)

The moral of example (h): You can prove anything from a contradiction. So you can get whatever you want from a contradiction.

2. How to use subproofs correctly?

2.1 The structure of a subproof

\[ \begin{array}{c}
P \quad \text{the given premises} \\
\vdots \end{array} \]

\[ \begin{array}{c}
Q \quad \text{the assumption of a subproof} \\
\vdots \end{array} \]

\[ \begin{array}{c}
\text{R} \\
\vdots \end{array} \]

\[ \begin{array}{c}
\text{S} \\
\vdots \end{array} \]

\[ \text{a closed subproof} \]

\[ \begin{array}{c}
\text{T} \quad \text{C} \\
\vdots \end{array} \]

\[ \begin{array}{c}
\text{U} \\
\vdots \end{array} \]

\[ \begin{array}{c}
\text{A} \\
\vdots \end{array} \]

\[ \begin{array}{c}
\text{B} \\
\vdots \end{array} \]

\[ \text{W} \]

\[ \text{a closed sub-subproof} \]

\[ \text{A closed subproof} \]

“C” is what you want to prove (the logical consequence of \( P \)) which is always at the bottom of the main proof and outside any subproof.

2.2 Some features of a subproof

- A subproof begins with an unproved assumption, which can only be used inside the subproof itself, and cannot be used outside a closed subproof.
- Once a subproof has been closed off, it can only be cited as a whole. Its individual items are not available anymore.
- A subproof can cite items that occur earlier outside the subproof, so long as they do not occur in another subproof that have been closed off.
2.3 An example

Example (i): \(\{(B \land A) \lor (A \land C)\} \vdash A \land B \ ??

1. \((B \land A) \lor (A \land C)\)

\[\begin{align*}
2. & \quad B \land A \\
3. & \quad B \quad \land \text{Elim: 2} \\
4. & \quad A \quad \land \text{Elim: 2} \\
5. & \quad A \land C \\
6. & \quad A \quad \land \text{Elim: 5} \\
7. & \quad A \quad \lor \text{Elim: 1, 2-4, 5-6} \\
8. & \quad A \land B \quad \land \text{Intro: 7, 3} \quad \text{WRONG!!!} \quad \text{“B” is inside a closed off subproof which cannot be used outside.}
\end{align*}\]

3. Examples of proofs

3.1 Some tips of proofs

- If premises contain a disjunction, try “\(\lor\)Elim”.
- If premises contain no disjunction, try “\(\neg\)Intro” by negating the conclusion.
- From a contradiction, you can get whatever you want by using “\(\neg\)Intro”.

Suppose that during a proof, you disparately need “\(R\)”. Fortunately, there is a contradiction occurring earlier. Then you can get “\(R\)” by doing the following:

\[\begin{align*}
&\vdash P \\
&\vdash \neg P \\
&\vdash \neg R \\
&\vdash P \land \neg P \\
&\quad \vdash \neg R \\
&\vdash R
\end{align*}\]

- Making use of Quasi-disjunction to help to get a contradiction.

\[\begin{align*}
&\neg (P \lor Q) \\
&\quad \vdash P \\
&\quad \vdash P \lor Q \\
&\quad (P \lor Q) \land \neg (P \lor Q) \\
&\quad \neg P \\
&\quad \vdash Q \\
&\quad \vdash P \lor Q \\
&\quad (P \lor Q) \land \neg (P \lor Q) \\
&\quad \neg Q \\
&\quad \neg Q \land \neg P
\end{align*}\]
3.2. More examples

Example (j): \{ \phi \} \models \neg (P \land Q \land \neg P)

1. \phi  
   tip: try “\neg Intro”

2. P \land Q \land \neg P
3. P \land \neg P \land Elim: 3
4. \neg (P \land Q \land \neg P) \land Intro: 3-4

Example (k): \{ \neg P \lor \neg Q \} \models \neg (P \land Q)  
DeMorgan Rule

1. \neg P \lor \neg Q  
   tip: try “\neg Intro” first.

2. P \land Q  
   tip: introduce more information

3. \neg P \land Elim: 2

4. P \land Elim: 2
5. P \land \neg P \land Intro: 3, 4

6. \neg Q
7. Q \land Elim: 2

8. \neg (P \land Q) \land Intro: 6, 7
9. Q \land \neg Q \land Intro: 6, 7

10. \neg (P \land \neg P) \land Intro: 8-9

11. P \land \neg P \land Elim: 10

12. P \land \neg P \lor \land Elim: 1, 3-5, 6-11

13. \neg (P \land Q) \land \neg Intro: 2-12

Example (l): \{ \neg (P \land \neg R) \} \models \neg P \lor \neg \neg R  
DeMorgan Rule

1. \neg (P \land \neg R)  

2. \neg (\neg P \lor \neg R)  
   tip: try “\neg Intro” first (goal: try to get a contradiction)

3. \neg P
4. \neg P \lor \neg R \lor \land Intro: 3
5. (\neg P \lor \neg R) \land \neg (\neg P \lor \neg R) \land Intro: 2, 4
6. \neg \neg P \land \neg Intro: 3-5

7. P  
   \land Elim: 6

8. \neg R
9. \neg P \lor \neg R \lor \land Intro: 8

10. (\neg P \lor \neg R) \land \neg (\neg P \lor \neg R) \land Intro: 2, 9
11. \neg \neg R \land \neg Intro: 8-10
12. R  
   \land Elim: 11
13. P \land R \land Intro: 7, 12
14. (P \land R) \land \neg (P \land R) \land Intro: 13, 1

15. \neg (\neg P \lor \neg R) \land \neg Intro: 2-14
16. P \lor \neg R \land \neg Elim: 15
1.4. Notes for Chapter 4: Conditionals and Biconditionals

1.4.1. Material Conditional/Biconditional Symbols
(4.1-4.2 of the Text)

1. Syntax of material conditionals/biconditionals

<table>
<thead>
<tr>
<th>Conditionals</th>
<th>English</th>
<th>FOL</th>
</tr>
</thead>
<tbody>
<tr>
<td>If-then</td>
<td>If P, then Q.</td>
<td>P → Q</td>
</tr>
<tr>
<td></td>
<td>Q, if P.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Q, provided P.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Q, assuming that P.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Q when P.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Q just in case that P</td>
<td></td>
</tr>
<tr>
<td>Only if</td>
<td>P, only if Q = If P, then Q.</td>
<td>P → Q</td>
</tr>
<tr>
<td>Unless</td>
<td>P unless Q = If not Q, then P.</td>
<td>P ∨ Q ↔ ¬ Q → P ↔ ¬P → Q</td>
</tr>
<tr>
<td></td>
<td>“unless” = “if not”</td>
<td></td>
</tr>
<tr>
<td>Biconditionals</td>
<td>P if and only if Q.</td>
<td>P ↔ Q ↔ (P → Q) ∧ (Q → P)</td>
</tr>
</tbody>
</table>

2. Necessary and sufficient conditions

2.1 Definitions

Event A is a sufficient condition of event B iff whenever A occurs B will necessarily follow.

Event A is a necessary condition of event B iff in order for B to occur, A has to be occur first, or, whenever B occurs A necessarily occurs.

2.2 Examples

Usually, we use “if-then” to express a sufficient condition (a sufficient condition follows “if”):
- If it rains outside, then the (uncovered) ground outside will be wet.
- If my car runs, then there is enough gas in the tank.
- If someone is the president of US, then he or she is over 35 years old.

Accordingly, we use “only if” to express a necessary condition (a necessary condition follows “only if”):
- It rains outside only if the (uncovered) ground outside is wet.
- A car can run only if there is enough gas in the tank.
- A person can become the president of US only if he or she is over 35 years old.
2.3 Relation between sufficient and necessary conditions

If A is a sufficient condition of B, then B is a necessary condition of A.

\[ A \rightarrow B \]

A is a sufficient condition of B
B is a necessary condition of A

A sufficient condition is on the left of “→”
A necessary condition is on the right of “→”

3. Logical rules

Contraposition

\[ P \rightarrow Q \iff \neg Q \rightarrow \neg P \]

Rules of Inference

\[
\begin{array}{c}
P \rightarrow Q \\
P \\
\hline
Q \\
\end{array} \quad \text{Modus Ponens (MP)}
\]

\[
\begin{array}{c}
P \rightarrow Q \\
\neg Q \\
\hline
\neg P \\
\end{array} \quad \text{Modus Tollens (MT)}
\]

4. Semantics of material conditionals/biconditionals

4.1 Truth-table definition

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P → Q</th>
<th>P ↔ Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

4.2 Material conditional is truth-functional connective

Material conditionals === truth-functional conditionals

Although our definition of “→” matches the meanings of some conditional statements in ordinary language very well, the match is far from perfect. For example, the following statements seem to express some meanings which cannot be expressed fully by our definition of “→”.

(1) Conditionals used in the subjective mood (counterfactual / subjective conditionals)

If Max had been at home (the antecedent is in fact false), then Carl would have been there too (the conditional sentence might be true).

If President Kennedy had not been assassinated (the antecedent is in fact false), he would have been reelected (the conditional sentence may be true).

If I jumped out of the window, then I would become a butterfly.
If I were the president of US (the antecedent is in fact false), then I would be a most powerful man on earth (the conditional sentence is true).

If the Washington Monument were made of lead (the antecedent is in fact false), then it would be lighter than air (the conditional sentence is false).

We usually call these conditionals counterfactuals because their antecedents are typically false. However, the truth values of counterfactual conditionals are not fully determined by the truth values of component sentences. The only way of determining their truth values is through some kind of inference based on our common knowledge.

(2) Conditionals used in the indicative mood with inferential connection between antecedent and consequent.

- If the temperature rises above 32F, then the snow will begin to melt.
- If figure A is a triangle, then figure A has three sides.
- If all A are B and all B are C, then all A are C.

For these conditionals which express the genuine inferential connection between antecedent and consequent, they remains true even though their antecedent might be false.

In conclusion, “→” does not express any inferential connection between antecedent and consequent, but only express the relation between the truth value of conditionals and that of their antecedents and consequents. Therefore, since many conditionals sentences in ordinary language express a inferential connection, when “→” is used to translate them, part of their meaning is left out.
1.4.2. Formal Proofs Involving the Conditional (4.5 of the Text)

1. Derivation rules for the conditional and the biconditional

**Conditional Rules**

| #m. P → Q          | #m. P     |
|                   |           |
| :                 | #n. Q     |
| Q → Elim: #m, #n  |           |

**Biconditional Rules**

| #m P ↔ Q            | #m P     |
|                     |           |
| :                  | #n Q     |
| Q / P ↔ #m, #n     |           |

2. Tips of proofs

- If the conclusion is conditional or biconditional, then try "→Intro" or "↔Intro".
- If the conclusion is not conditional, try "¬Intro".
- If one premise is disjunction, try "∨Elim".

3. Class exercises

**Example 1** \{(A ∨ B) → C\} ⊨ A → C

1. (A ∨ B) → C
2. A
3. A v B v Intro: 2
4. C → Elim: 1, 3
5. A → C → Intro: 2-4

**Example 2** \{φ\} ⊨ A → ¬¬A

1. φ
2. A
3. ¬A
4. A ∧ ¬A ∧ Intro: 2, 3
5. ¬¬A ¬¬Intro: 3-4
6. A → ¬¬A → Intro: 2-5
**Example 3** (problem 23-5) \( \{A \leftrightarrow B, B \leftrightarrow C\} \vdash C \leftrightarrow A \)

1. \(A \leftrightarrow B\)
2. \(B \leftrightarrow C\)
3. \(C\)
4. \(B \leftrightarrow\) Elim: 2, 3
5. \(A \leftrightarrow\) Elim: 1, 4
6. \(A\)
7. \(B \leftrightarrow\) Elim: 1, 6
8. \(C \leftrightarrow\) Elim: 2, 7
9. \(A \leftrightarrow C\) \(\leftrightarrow\) Intro: 3-5, 6-8

**Example 4** \(\{\neg P \lor Q\}\) \(\vdash P \rightarrow Q\)

1. \(\neg P \lor Q\)
2. \(P\)
3. \(Q\)
4. \(\neg P\) \(\leftrightarrow\) Intro: 3
5. \(\neg P\)
6. \(P \land \neg P\) \(\land\) Intro: 2, 5
7. \(P \land \neg P\)
8. \(\neg Q\) \(\neg\) Intro: 6-7
9. \(Q\) \(\neg\) Elim: 8
10. \(Q\) \(\lor\) Elim: 1, 3-4, 5-9
11. \(P \rightarrow Q\) \(\rightarrow\) Intro: 2-10

**Example 5** \(\{P \rightarrow Q\}\) \(\vdash \neg P \lor Q\)

1. \(P \rightarrow Q\)
2. \(\neg (\neg P \lor Q)\)
3. \(\neg P\)
4. \(\neg P \lor Q\) \(\lor\) Intro: 3
5. \(\neg P \lor Q\)
6. \(\neg P\) \(\neg\) Intro: 3-5
7. \(P\) \(\neg\) Elim: 6
8. \(Q\) \(\rightarrow\) Elim: 1, 7
9. \(Q\)
10. \(\neg P \lor Q\) \(\lor\) Intro: 9
11. \(\neg P \lor Q\)
12. \(\neg Q\) \(\neg\) Intro: 9-11
13. \(Q \land \neg P\) \(\land\) Intro: 8, 12
14. \(\neg (\neg P \lor Q)\) \(\neg\) Intro: 2-13
15. \(\neg P \lor Q\) \(\neg\) Elim: 14
Example 6 \{A \lor B, \neg B, A \rightarrow C\} \vdash C

1. A \lor B
2. \neg B
3. A \rightarrow C

4. A

5. A \hspace{1cm} \text{Reit: 4}

6. B

7. \neg A

8. B \land \neg B \hspace{1cm} \land \text{Intro: 6, 2}

9. \neg A \hspace{1cm} \neg \text{Intro: 7-8}

10. A \hspace{1cm} \neg \text{Elim: 9}

11. A \hspace{1cm} \lor \text{Elim: 1, 4-5, 6-10}

12. C \hspace{1cm} \rightarrow \text{Elim: 3, 11}

Example 7 \{P \rightarrow Q, \neg Q, \neg P \rightarrow R\} \vdash R \lor Q

1. P \rightarrow Q

2. \neg Q

3. \neg P \rightarrow R

4. P

5. Q \hspace{1cm} \rightarrow \text{Elim: 1, 4}

6. Q \land \neg Q \hspace{1cm} \land \text{Intro: 5, 2}

7. \neg P \hspace{1cm} \neg \text{Intro: 4-6}

8. R \hspace{1cm} \rightarrow \text{Elim: 3, 7}

9. R \lor Q \hspace{1cm} \lor \text{Intro: 8}
1.5. Notes for Chapter 5: Introduction to Quantification

1.5.1. Basic Components of FOL (5.1-5.4 of the Text)

1. Quantifiers introduced
How can we translate the following sentences into logical notation?

"Some block is cube."
"All blocks are cube."
"Only one block is cube."
"There are two cubes."
"Every/each cube is large."
"There is a cube which is larger than any other blocks."
"At least two blocks are cube."
"At most two blocks are cube."

We notice that these sentences contain so-called determiners used to express the quantity of objects:

Universal: all, every, each, any
Existential: a/an, some, at least one, “any”
Numbers: one, two, three, ...
Definite description: the (present king of France)
Others: many, almost all, a few

To symbolize these determiners, we need to introduce quantifier symbols.

Definition: quantifiers are logical symbols used to indicate the quantity of objects in a category.

Universal quantifier: $\forall x$ read as "For every object $x$ in the domain D, $x$ ...
Existential quantifier: $\exists x$ read as "For at least one object $x$ in the domain D, $x$ ...

Examples:
“Some block is cube.” $\exists x \text{Cube}(x)$ “For at least one block $x$, $x$ is a cube.”
“Some cubes are large.” $\exists x (\text{Cube}(x) \land \text{Large}(x))$ “For at least one cube $x$, $x$ is large.”
“All blocks are cube.” $\forall x \text{Cube}(x)$ “For every block $x$, $x$ is a cube.”
“All cubes are large.” $\forall x (\text{Cube}(x) \rightarrow \text{Large}(x))$ “For every cube $x$, $x$ is large.”
2. Review: the components of FOL

<table>
<thead>
<tr>
<th>Definitions</th>
<th>English Expressions</th>
<th>FOL</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Descriptive Symbols</strong> (without fixed meaning)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Terms:</strong> referring to one or more individual objects.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>• <strong>Names:</strong> referring to fixed individuals.</td>
<td>Max, 2:00 pm, music #2, happiness</td>
<td>the same</td>
</tr>
<tr>
<td>• <strong>Variables:</strong> referring to unfixed individuals</td>
<td>some block, a person</td>
<td>x, y, z</td>
</tr>
<tr>
<td>• <strong>Functions:</strong> referring to either fixed or unfixed individuals.</td>
<td>Max's father, someone's father</td>
<td>father(Max), father(x)</td>
</tr>
<tr>
<td><strong>Predicates:</strong> denoting properties of objects or relations between objects</td>
<td>x is red, x loves y</td>
<td>Red(x), Love(x, y)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Logical Symbols</th>
<th>with fixed meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sentential connectives:</strong> connecting sentences.</td>
<td>and, or, not, if-then, iff</td>
</tr>
<tr>
<td><strong>Quantifiers:</strong> indicating the quantity of objects in a category.</td>
<td></td>
</tr>
<tr>
<td>• <strong>Universal quantifiers</strong></td>
<td>all, every, each, any</td>
</tr>
<tr>
<td>• <strong>Existential quantifiers</strong></td>
<td>a, some, at least one</td>
</tr>
</tbody>
</table>

3. Wffs and sentences

3.1 The scope of a quantifier

Rule: like “¬”, a quantifier symbol only applies to the smallest unite after it.

\[
\forall x \ (\text{Doctor}(x) \rightarrow \text{Rich}(x)) \\
\text{the scope of } \forall x
\]

\[
\forall x \text{ Doctor}(x) \rightarrow \text{Rich}(x) \\
\text{the scope of } \forall x
\]

\[
\exists x \ (\text{Doctor}(x) \land \text{Rich}(x)) \\
\text{the scope of } \exists x
\]

\[
\exists x \text{ Doctor}(x) \land \text{Rich}(x) \\
\text{the scope of } \exists x
\]

\[
\exists x \forall y \ (\text{Love}(x, y) \land \text{Love}(y, x)) \\
\text{the scope of } \exists x \text{ and } \forall y
\]

\[
\exists x \ (\text{Love}(x, y) \land \forall y \text{ Love}(y, x)) \\
\text{the scope of } \exists x
\]
3.2 Free and bound variables

An occurrence of a variable x/y is free if it is outside of the scope of the corresponding quantifier.

An occurrence of a variable x/y is bound if it is inside of the scope of the corresponding quantifier.

3.3 wffs

Atomic formula: a formula containing one predicate only, such as Home(x), Love(x, Max), x = Max

A formula is well formed, which is called wff, iff its formulation of is syntactically correct (proper use of parentheses, logical symbols, and descriptive symbols) as defined on p. 117 (Rules 1-5).

3.4 Sentences

A formula is a sentence iff it contains no free variables.

Formula

Well-formed formula

Sentence: ∀x (Doctor(x) → Rich(x))

Non-sentence: ∀x Doctor(x) → Rich(x)

Not well-formed formula:

∀ (Doctor(x) → Rich(y))

Missing x in a quantifier symbol

∀x (Doctor(y) → Rich(y))

No matching variable identified by ∀x

∀x (Doctor(x) → Rich(x) ∧ Selfish(x))

Missing “( )”
1.5.2. Semantics for the Quantifiers (5.5 of the Text)

1. Truth-values of wffs?

   Only a sentence can have a truth-value (either true or false). So a non-sentence wff has no truth value whatsoever.

   Love (x, Max) ???

2. Satisfaction of wffs

   Definition: Suppose a non-sentence wff S(x). If for a given object b such that S(b) is true, then we say that b satisfies S(x).

   Examples:
   • For “Cube(x)”, if b is a cube, then b satisfies “Cube(x)”.
   • For “Doctor(x) ∧ Rich(x)”, if a person p is both a doctor and rich, then p satisfies “Doctor(x) ∧ Rich(x)”.

   To say a wff S(x) is satisfiable iff there exists at least one object in the domain of discourse which can satisfy S(x), that is, which can make S(x) true.

   For example, to say that “Cube(x)” is satisfiable in the world if there exists a cube in the world.

3. Domain of discourse

   Whether a wff is satisfiable depends on the domain of discourse, that is, in which area we suppose to locate an object which satisfies the wff. For example, “Female(x) ∧ Student(x)” is satisfiable in the domain of all the people in this classroom, but it is not satisfiable if we restrict our discourse on a classroom of Deep Spring College at CA (the whole college consists of 26 young men). Similarly, “Doctor(x) ∧ Rich(x)” is not satisfiable in the domain of this classroom. But it is satisfiable in the area of Huntingdon County.

   The domain of discourse D is the intended category / collection of all objects under discussion.

   Usually, we use the symbol D to represent a domain, such as:
   D: all things in the universe.
   D: all people in the world.
   D: all cats in a city.
   D: all students in this room.
   D: all blocks in a Tarski’s world.

4. Variables and domain

   Accordingly, a variable of a quantifier, such as x in ∀x or ∃x, is supposed to range over all the objects in a domain, or is supposed to refer to any or some object in a domain. For example, suppose that D: all people in this classroom. Then x in the sentence “∃x (female(x) ∧ Student(x))” ranges over all people in the room or refers to some people in the room. For the same sentence, suppose that D: all things in Juniata College. Then x in “∃x (female(x) ∧ Student(x))” ranges over all things in JC or refers to some thing in JC.
5. **Truth values of sentences**

A universally quantified sentence $\forall x \, S(x)$ is true iff $S(x)$ is satisfied by every object in the domain.

A existentially quantified sentence $\exists x \, S(x)$ is true iff $S(x)$ is satisfied by at least one object in the domain.

Suppose D: all people in this room.

| $\exists x \, \text{Girl}(x)$ | True | (Someone is a girl) |
| $\forall x \, \text{Girl}(x)$ | False | (Everyone is a girl) |
| $\exists (\text{Girl}(x) \land \text{StudyPhil}(x))$ | False | (Some girl studies philosophy) |
| $\exists (\text{Father}(x) \land \text{Doctor}(x))$ | ? | (Some father is a doctor) |
| $\forall (\text{Mother}(x) \rightarrow \text{Lawyer}(x))$ | False | (Every mother is a lawyer) |

Suppose D: all blocks in a Tarski’s world.

......

6. **Domain, translation, truth values**

Obviously, whether a quantified sentence is true depends partially upon the domain of discourse. As we will find out shortly, the choice of domain of discourse plays an essential role in translation also.

<table>
<thead>
<tr>
<th>Domain</th>
<th>English Sentence</th>
<th>Translation</th>
<th>Truth value</th>
</tr>
</thead>
<tbody>
<tr>
<td>All people in JC</td>
<td>Some students are female.</td>
<td>$\exists x , (\text{Student}(x) \land \text{Female}(x))$</td>
<td>True</td>
</tr>
<tr>
<td>All students in JC</td>
<td>Some students are female.</td>
<td>$\exists x , \text{Female}(x)$</td>
<td>True</td>
</tr>
<tr>
<td>All students in DSC</td>
<td>Some students are female.</td>
<td>$\exists x , \text{Female}(x)$</td>
<td>False</td>
</tr>
<tr>
<td>All people in JC</td>
<td>All students are male.</td>
<td>$\forall x , (\text{Student}(x) \rightarrow \text{Male}(x))$</td>
<td>False</td>
</tr>
<tr>
<td>All students in JC</td>
<td>All students are male.</td>
<td>$\forall x , \text{Male}(x)$</td>
<td>False</td>
</tr>
<tr>
<td>All people in DSC</td>
<td>All students are male.</td>
<td>$\forall x , (\text{Student}(x) \rightarrow \text{Male}(x))$</td>
<td>True</td>
</tr>
</tbody>
</table>
1.5.3. Translation of Sentences with Quantifiers (5.7-5.8 of the Text)

Part 1: Introduction to A/E/I/O sentences

1. Standard forms of A, E, I, O sentences

<table>
<thead>
<tr>
<th>English sentences</th>
<th>Sentence forms</th>
<th>Quantity</th>
<th>Quality</th>
</tr>
</thead>
<tbody>
<tr>
<td>All cube are small.</td>
<td>All S are P</td>
<td>universal</td>
<td>affirmative</td>
</tr>
<tr>
<td>No cube is small.</td>
<td>No S is P</td>
<td>universal</td>
<td>negative (complete)</td>
</tr>
<tr>
<td>Some cube is small.</td>
<td>Some S is P</td>
<td>particular</td>
<td>affirmative</td>
</tr>
<tr>
<td>Some cube is not small.</td>
<td>Some S is not P</td>
<td>particular</td>
<td>negative (partial)</td>
</tr>
</tbody>
</table>

Common pattern:

Determiners + Noun phrase + (Negation) + Linking verb + Predicate (adjective/noun)

2. Quasi-forms of A, E, I, O sentences

Many quantified English sentences without standard quantity terms can be paraphrased as standard A/E/I/O/ sentences.

- Some dogs swim. ⇔ Some dogs are creatures which swim.
- Each/Every/Any dog is friendly. ⇔ All dogs are friendly.
- Only cubes are small. ⇔ All small blocks cube.
- The only cubes are small. ⇔ All cubes are small.
- A cube is small. ⇔ Some cube is small.
- A cube is a block with six equal square surfaces. ⇔ All cubes are blocks with six equal square surfaces.

3. Square of opposition

A: All S are P.       E: No S is P.
I: Some S is P.       O: Some S is not P.

Two sentences are contradictory iff they have opposite quantity and quality.
Part 2: Translation Drills

Notes:
- If not specified, the domain of discourse (DD) of the following translations will be: All things in the world or all blocks in a Tarski's World.
- Sentences with "*" are sentences which we will study further later.
- "¬" means "¬".

1. Standard A/E/I/O Sentences

A: All (All the / Every / Each) S are / is P,  \[ \forall x \ (S(x) \rightarrow P(x)) \iff \neg \exists x \ (S(x) \land \neg P(x)) \]

All cubes are small.  \[ \forall x \ (\text{Cube} \ (x) \rightarrow \text{Small} \ (x)) \]
All cubes to the right of a are small.  \[ \forall x \ [(\text{Cube} \ (x) \land \text{RightOf} \ (x, a)) \rightarrow \text{Small} \ (x)] \]
Some cube is to the right of any small cubes.  \[ \exists x \ {\text{Cube}(x) \land \forall y \ [(\text{Cube}(y) \land \text{Small} \ (y) \land x \neq y) \rightarrow \text{RightOf}(x, y)] } \]

All pussy cats are cute  \[ \forall x \ [(\text{Cat} \ (x) \land \text{Pussy} \ (x)) \rightarrow \text{Cute} \ (x)] \]
All cats which Jenny likes are cute.  \[ \forall x \ [ (\text{Cat} \ (x) \land \text{Like} \ (\text{Jenny}, x)) \rightarrow \text{Cute} \ (x)] \]
Everyone whom Jenny likes dislikes Jenny.  \[ \forall x \ [(\text{Person}(x) \land \text{Like}(\text{Jenny}, x)) \rightarrow \text{Dislike}(x, \text{Jenny})] \]
Michael likes everyone that Jenny or Ron hates.  \[ \forall x \ [ (\text{Person}(x) \land (\text{Hate} \ (\text{Jenny}, x) \lor \text{Hate} \ (\text{Ron}, x))) \rightarrow \text{Like} \ (\text{Michael}, x)] \]

E: No S (None of the S) are / is P,  \[ \forall y \ (S \ (y) \rightarrow \neg P(y)) \iff \neg \exists y \ (S(y) \land P(y)) \]

No cubes are small.  \[ \forall y \ (\text{Cube} \ (y) \rightarrow \neg \text{Small} \ (y)) \]
No cats which Jenny likes are the cats that eat birds*.  \[ \forall x \ [(\text{Cat} \ (x) \land \text{Like} \ (\text{Jenny}, x)) \rightarrow \neg \exists y \ (\text{Bird} \ (y) \land \text{Eat} \ (x, y))] \]
Jenny likes no philosopher whom Michael likes.  \[ \forall x \ [(\text{Phil} \ (x) \land \text{Like} \ (\text{Michael}, x)) \rightarrow \neg \text{Like} \ (\text{Jenny}, x)] \]

I: Some S (Some of the S) are is P.  \[ \exists x \ (S(x) \land P(x)) \iff \neg \forall x \ (S(x) \lor \neg P(x)) \]

Some cubes are small.  \[ \exists x \ (\text{Cubes} \ (x) \land \text{Small} \ (x)) \]
Some small cubes is in front of b.  \[ \exists x \ (\text{Cubes} \ (x) \land \text{Small} \ (x) \land \text{FrontOf} \ (x, b)) \]
Some cube is to the right of a small tetrahedron*. \[ \exists x \ [\text{Cubes} \ (x) \land \exists y \ (\text{Small} \ (y) \land \text{Tet}(y) \land \text{RightOf} \ (x, y))] \]
Some person whom Jenny likes studies philosophy.

$\exists x \ [\text{Person}(x) \land \text{Like}(Jenny, x) \land \text{StudyPhil}(x)]$

O. Some S are / is not P.

$\exists x \ (S(x) \land \sim P(x)) \iff \sim \forall x \ (S(x) \rightarrow P(x))$

Some cubes are not small.

$\exists x \ (\text{Cube}(x) \land \sim \text{Small}(x))$

Some gray bugs are not what Jenny likes.

$\exists x \ [\text{Green}(x) \land \text{Bug}(x) \land \sim \text{Like}(Jenny, x)]$

Jenny does not like some girls whom Michael likes.

$\exists x \ [\text{Girl}(x) \land \text{Like}(Michael, x) \land \sim \text{Like}(Jenny, x)]$

Jenny does not like someone who is a philosopher.

$\exists x \ [\text{Phil}(x) \land \sim \text{Like}(Jenny, x)]$

2. Some Quasi-A/E/I/O Sentences

2.1 "a" (context-sensitive)

There is a bird in this room $\iff$ Some bird is in this room.

$\exists x \ (\text{Bird}(x) \land \text{InRoom}(x))$

A bird is not warm-blooded. $\iff$ All birds are not warm-blooded.

$\forall x \ (\text{Bird}(x) \rightarrow \sim \text{Warm}(x))$

A cube is small.

$\exists x \ (\text{Cube}(x) \land \text{Small}(x))$

2.2 "The" (context-sensitive)

The whale is mammal. $\iff$ All whales are mammals.

$\forall x \ (\text{Whale}(x) \rightarrow \text{Mammal}(x))$

The whale (refer to the one we are talking about) is gone*.

$\exists x \ [\text{Whale}(x) \land \text{Gone}(x) \land \forall y \ (\text{Whale}(y) \rightarrow x = y)]$

The cube is small*.

$\exists x \ [\text{Cube}(x) \land \text{Small}(x) \land \forall y \ (\text{Cube}(y) \rightarrow y = x)]$

2.3 No quantity terms at all (context-sensitive)

Dogs are smart. $\iff$ All dogs are smart.

$\forall x \ (\text{Dog}(x) \rightarrow \text{Smart}(x))$

Dogs are present. $\iff$ Some dogs are present.

$\exists x \ (\text{Dog}(x) \land \text{Present}(x))$

2.4 "any" (context-sensitive)

The quantity term "any" has either the force of "every" or that of "some" depending on the context.
(1) "Any" as "every" (Tip: “any” as subject or as predicate of a negative sentence)

Anything that tastes good is bad. ⇔ All things that taste good are bad.
∀x (TasteG(x) → Bad(x))

Anyone who fails the final flunks the course.  ∀x [(Person(x) ∧ Fail(x)) → FlunkC(x)]

Jenny does not like any bugs. ⇔ Jenny likes no bugs.  ∀x (Bug (x) → ~ Like (Jenny, x))

Jenny likes anyone.  ∀x (Person(x) → Like(Jenny, x))

(2) "Any" as "some" (Tip: “any” inside the antecedent of a conditional)

If anybody loves Jenny, then Tom does. ⇔ Tom, if anyone, loves Jenny. ⇔ If someone loves Jenny, then Tom does.
∃x (Person(x) ∧ Like(x, Jenny)) → Like(Tom, Jenny) ⇔
∀x [(Person(x) ∧ Like(x, Jenny)) → Like(Tom, Jenny)]

• Comparison: If everyone loves Jenny, then Tom does.
∀x (Person(x) ∧ Like(x, Jenny)) → Like(Tom, Jenny) ⇔ ∃x [(Person(x) ∧ Like(x, Jenny)) → Like(Tom, Jenny)]

(3) mixed case

Anyone (as "everyone") who loves anyone (as "someone") is loved by someone.*
∀x [∃y (Person(x) ∧ Like(x, y)) → ∃z (Person(z) ∧ Like(z, x))]

2.5 "All ... not" (context-sensitive)

All that glistens is not gold. ⇔ Not all that glistens is gold (partial denial).

∼ ∀x (Glisten(x) → Gold(x))

All cubes are not tetrahedrons. ⇔ No cubes are tetrahedrons (complete denial).
∀x (Cube(x) → ∼ Tet(x))

3. Comparison

DD: All people in JC.

a is largest cube.  Cube(a) ∧ ∀x [(Cube(x) ∧ x ≠ a) → Larger(a, x)]

a is as small as b.  ∼ Smaller(a, b) ∧ ∼ Smaller(b, a)

Michael is as tall as Tom.  ∼ Taller(Tom, Michael) ∧ ∼ Taller(Michael, Tom)

Michael is the tallest boy.  Boy(Michael) ∧ ∀x [(Boy(x) ∧ x ≠ Michael) → Taller(Michael, x)]

Jenny is the shortest girl.  Girl(Jenny) ∧ ∀x [(Girl(x) ∧ x ≠ Jenny) → Taller(x, Jenny)]

Jenny is the eldest daughter of Ron.
(Jenny = daughter(Ron)) ∧
∀x [(x = daughter(Ron) ∧ x ≠ Jenny) → Older(Jenny, x)]
4. Exclusive Sentences

• "Only S are P" claims that "to be S" is necessary, but not sufficient, for "to be P".
• Here the term that follows "only" is a general term "S" (a plural noun or pronoun). For the case of a singular term, see later "numbers" section.

4.1 Only (None but) S are P. \(\iff\) All P are S. \(\forall x \ (P(x) \to S(x))\)

- Only cubes are small. \(\iff\) All small blocks are cubes. \(\forall x \ (\text{Small}(x) \to \text{Cube}(x))\)
- Only cubes are to the right of \(b\). \(\forall x \ (\text{RightOf}(x, b) \to \text{Cube}(x))\)
- Only citizens can vote. \(\iff\) All those who can vote are citizens (wrong paraphrase: All citizens can vote). \(\forall x \ (\text{Vote}(x) \to \text{Citizen}(x))\)

A Tiger eats only meat. \(\iff\) All foods that a tiger eats are meat.* \(\forall x \ [\text{Food}(x) \to \forall y ((\text{Tiger}(y) \land \text{Eat}(y, x)) \to \text{Meat}(x))]\)

Claire had only blank disks at 3:00 pm. \(\forall x \ [(\text{Disk}(x) \land \text{Owned}(\text{Claire}, x, 3:00)) \to \text{Blank}(x, 3:00)]\)

- \(a\) is only to the right of some cubes. \(\forall x \ (\text{RightOf}(a, x) \to \text{Cube}(x))\)

4.2 The only S are P. \(\iff\) All S are P. \(\forall x \ (S(x) \to P(x))\)

- Reading is the only thing that Jenny enjoys to do (The only thing that Jenny enjoys to do is reading). \(\forall x \ (\text{Enjoy}(\text{Jenny}, x) \to \text{Read}(x))\)

• Comparison:

  Only students in this room are logic students. \(\Rightarrow\) All logic students are the students in this room (implication: there may be some other non-logic students in this room; but no logic students are outside the room).
  \(\forall x \ [(\text{Student}(x) \land \text{Logic}(x)) \to \text{InRoom}(x)]\)

  The only students in this room are logic students. \(\Rightarrow\) All students in this room are logic students (implication: There may be some logic students outside this room; but there is no other non-logic students in this room; of course, there may be some non-students, such as an instructor, in the room).
  \(\forall x \ [(\text{Student}(x) \land \text{InRoom}(x)) \to \text{Logic}(x)]\)

  Only cubes are small. \(\iff\) All small blocks are cube (There may be other medium or large cubes; there may be non-cube blocks as long as they are not small; but there is no any other non-cube block which is small).
  \(\forall x \ (\text{Small}(x) \to \text{Cube}(x))\)

  The only cubes are small. \(\iff\) All cubes are small (There may be other shapes of blocks with different sizes, such as a small tetrahedron; but there is no cube which is not small).
  \(\forall x \ (\text{Cube}(x) \to \text{Small}(x))\)

DD: All people in JC.

Rule: Where a quantity term ("a", "any", "some", etc.) is used in the antecedent of an English conditional and there is, in the consequent of the conditional, pronominal cross-reference to that quantity term, a universal quantifier is called for.

If anyone / someone / everyone loves Jenny, then that person loves Rita. ⇔ Anyone who loves Jenny loves Rita.

∀x ( Love (x, Jenny) → Love (x, Rita))

If someone fails the final, then the person cannot graduate this fall. ⇔ Anyone who fails the final cannot graduate this fall.

∀x ( Fail (x) → ~ Graduate (x))

If something is a cube, then it is large. ⇔ Anything which is a cube is large.

∀x ( Cube(x) → Large(x))

Every farmer who owns a donkey beats it*. ⇔ If a farmer owns a donkey, then the farmer beats the donkey.

∀x {Farmer (x) → ∀y [ (Donkey (y) ∧ Own (x, y)) → Beat (x, y)] }
⇔ ∀x {Farmer (x) → ∀y [ Donkey (y) → ( Own (x, y)) → Beat (x, y)]) } 
⇔ ∀x ∀y [ (Farmer (x) ∧ Donkey (y) ∧ Own (x, y) ) → Beat (x, y)]

Logical equivalence: (A ∧ B) → C ⇔ A → (B → C)

• For comparison:
If anybody loves Jenny, then Tom does. ⇒ If someone loves Jenny, then Tom does.
(without cross-reference)

∃x  Love (x, Jenny) → Love (Tom, Jenny) ⇔ ∀x ( Love (x, Jenny) → Love (Tom, Jenny) )

If Jenny loves someone, then Tom loves the same person (cross-reference).
∀x ( Love (Jenny, x) → Love (Tom, x) )

If anyone is rich, Bill is (without cross-reference)
∃x Rich (x) → Rich (Bill) ⇔ ∀x (Rich (x) → Rich (Bill))

If anyone is rich, it is Bill (cross-reference) ⇔ Only Bill is rich.
∀x (Rich (x) → x = Bill)
6. Complete and Partial Negation
DD: All people in JC.

6.1 Partial negation
O-sentences: Some S are not P:  \( \sim \forall x (S(x) \rightarrow P(x)) \leftrightarrow \exists x (S(x) \land \sim P(x)) \)

Not all students are females \( \leftrightarrow \) Some students are not females.  
\( \sim \forall x \ (\text{Student}(x) \rightarrow \text{Female}(x)) \leftrightarrow \exists x \ (\text{Student}(x) \land \sim \text{Female}(x)) \)

John does not like everyone.  \( \sim \forall x \ \text{Like}(John, x) \leftrightarrow \exists x \sim \text{Like}(John, x) \)

Not everyone likes John.  \( \sim \forall x \ \text{Like}(x, John) \leftrightarrow \exists x \sim \text{Like}(x, John) \)

6.2 Complete negation
E-sentences: No S are P:  \( \forall y (S(y) \rightarrow \sim P(y)) \leftrightarrow \sim \exists y (S(y) \land P(y)) \)

John does not like anyone (John likes nobody).  \( \sim \exists x \ \text{Like}(John, x) \)

Nobody likes John.  \( \sim \exists x \ \text{Like}(x, John) \)

1.5.4. Logical Equivalence Involving Quantifiers (5.9 of the Text)

1. Three pairs of logical equivalent sentences
To begin with, let us introduce three logical equivalences that we will use a lot in this section:

a.  \( P \rightarrow Q \leftrightarrow \sim P \lor Q \)

Proof:

1. \( P \rightarrow Q \)

<table>
<thead>
<tr>
<th>2. ( \sim (\sim P \lor Q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. ( \sim P )</td>
</tr>
<tr>
<td>4. ( \sim P \lor Q )</td>
</tr>
<tr>
<td>5. ( (\sim P \lor Q) \land \sim (\sim P \lor Q) )  ( \lor ) Intro: 3, 4</td>
</tr>
<tr>
<td>6. ( \sim \sim P )</td>
</tr>
<tr>
<td>7. ( P )</td>
</tr>
<tr>
<td>8. ( Q )</td>
</tr>
<tr>
<td>9. ( Q )</td>
</tr>
<tr>
<td>10. ( \sim P \lor Q )</td>
</tr>
<tr>
<td>11. ( (\sim P \lor Q) \land \sim (\sim P \lor Q) )  ( \land ) Intro: 10, 2</td>
</tr>
<tr>
<td>12. ( \sim Q )</td>
</tr>
<tr>
<td>13. ( Q \land \sim Q )</td>
</tr>
<tr>
<td>14. ( \sim (\sim P \lor Q) )</td>
</tr>
<tr>
<td>15. ( \sim P \lor Q )  ( \land ) Intro: 8, 12</td>
</tr>
<tr>
<td>16. ( \sim Q )</td>
</tr>
<tr>
<td>17. ( \sim P \lor Q )</td>
</tr>
<tr>
<td>18. ( \sim Q )</td>
</tr>
<tr>
<td>19. ( \sim P \lor Q )</td>
</tr>
<tr>
<td>20. ( Q \land \sim Q )</td>
</tr>
<tr>
<td>21. ( \sim Q )</td>
</tr>
<tr>
<td>22. ( \sim P \lor Q )</td>
</tr>
</tbody>
</table>

\( \land \) Intro: 2-13

\( \sim Q \)  |

\( \land \) Intro: 9-11

\( \sim Q \)  |

\( \land \) Intro: 8, 12

\( \sim Q \)  |

\( \land \) Intro: 2-13

\( \sim Q \)  |

\( \land \) Intro: 14.
and,

1. \(\neg P \lor Q\)
2. \(P\)
3. \(\neg P\)
4. \(\neg Q\)
5. \(P \land \neg P\)  \(\land\text{Intro: 2, 3}\)
6. \(\neg Q\)  \(\neg\text{Intro: 4-5}\)
7. \(Q\)  \(\neg\text{Elim: 6}\)
8. \(Q\)
9. \(Q\)
10. \(Q\)  \(\lor\text{Elim: 3-7, 8-9}\)
11. \(P \rightarrow Q\)  \(\rightarrow\text{Intro: 2-10}\)

b. \(\neg(P \rightarrow Q) \iff P \land \neg Q\)

Proof:
\(\neg(P \rightarrow Q) \iff \neg(\neg P \lor Q) \iff \neg P \land \neg Q \iff P \land \neg Q\)

c. \(P \rightarrow (Q \rightarrow R) \iff (P \land Q) \rightarrow R\)

Proof:
1. \(P \rightarrow (Q \rightarrow R)\)
2. \(P \land Q\)
3. \(P\)  \(\land\text{Elim: 2}\)
4. \(Q\)  \(\land\text{Elim: 2}\)
5. \(Q \rightarrow R\)  \(\rightarrow\text{Elim: 1, 3}\)
6. \(R\)  \(\rightarrow\text{Elim: 5, 4}\)
7. \((P \land Q) \rightarrow R\)  \(\rightarrow\text{Intro: 2-6}\)

and,

1. \((P \land Q) \rightarrow R\)
2. \(P\)
3. \(Q\)
4. \(P \land Q\)  \(\land\text{Intro: 2, 3}\)
5. \(R\)  \(\rightarrow\text{Elim: 1, 4}\)
6. \(Q \rightarrow R\)  \(\rightarrow\text{Intro: 3-5}\)
7. \(P \rightarrow (Q \rightarrow R)\)  \(\rightarrow\text{Intro: 2-6}\)
2. DeMorgan rules for quantifiers

According to DeMorgan rules, you can push “¬” past “∀ / ∃” as you can push a negation past a “(... ∧ ...) / (... ∨ ...)”.

\[
\neg \forall x \ P(x) \iff \exists x \ \neg P(x) \quad \forall x \ P(x) \iff \neg \exists x \ \neg P(x)
\]

\[
\neg \exists x \ P(x) \iff \forall x \ \neg P(x) \quad \exists x \ P(x) \iff \neg \forall x \ \neg P(x)
\]

Examples:

a. \(\neg \forall x (\text{Cube}(x) \to \text{Small}(x))\)  the negation of a A-sentence

\[
\iff \exists x \ (\neg (\text{Cube}(x) \to \text{Small}(x)))
\]

\[
\iff \exists x \ (\neg \text{Cube}(x) \lor \neg \text{Small}(x))
\]

\[
\iff \exists x \ (\text{Cube}(x) \land \neg \text{Small}(x))
\]

a corresponding O-sentence.

b. \(\neg \exists x (\text{Cube}(x) \land \text{Small}(x))\)  the negation of a I-sentence

\[
\iff \forall x \ (\neg \text{Cube}(x) \lor \neg \text{Small}(x))
\]

\[
\iff \forall x \ (\text{Cube}(x) \to \neg \text{Small}(x))
\]

3. Principle of replacing bound variables

Choice of a variable to stand for an unfixed individual is purely conventional. It does not matter which variable, \(x, y, z, w, \) or \(v\), that you choose to use, as long as the variable that is not already in use.

\[
\forall x \ (\text{Small}(x) \to \text{Cube}(x)) \iff \forall y \ (\text{Small}(y) \to \text{Cube}(y)) \iff \forall w \ (\text{Small}(w) \to \text{Cube}(w))
\]

For any wff \(P(x)\) without containing \(y\):

\[
\forall x \ P(x) \iff \forall y \ P(y) \quad \exists x \ P(x) \iff \exists y \ P(y)
\]

Examples:

a. \(\forall x ((\text{Small}(x) \land \text{Cube}(x)) \to \text{LeftOf}(x, a)) \iff \forall y ((\text{Small}(y) \land \text{Cube}(y)) \to \text{LeftOf}(y, a))\)

b. \(\forall x [(\text{Small}(x) \land \text{Cube}(x)) \to \exists y (\text{Tet}(y) \land \text{LeftOf}(x, y))]\)

\[
\iff \forall y [(\text{Small}(y) \land \text{Cube}(y)) \to \exists x (\text{Tet}(x) \land \text{LeftOf}(y, x))]
\]

\[
\iff \forall y [(\text{Small}(y) \land \text{Cube}(y)) \to \exists w (\text{Tet}(w) \land \text{LeftOf}(y, w))]
\]

but

\[
\forall y [(\text{Small}(y) \land \text{Cube}(y)) \to \exists y (\text{Tet}(y) \land \text{LeftOf}(y, y))] \text{ WRONG!!!}
\]

“\(y\)” occurs in the original formula.
4. Change of scopes of quantifiers
Suppose $A_x$ is any formula containing $x$, but $P$ is any formula that does not contain $x$ (but could contain other variables).

Rule 1: When $\exists x A_x$ or $\forall x A_x$ is the antecedent of a conditional and $P$ is a consequent of the conditional, then

$$
\exists x A_x \rightarrow P \iff \forall x (A_x \rightarrow P)
$$

$$
\forall x A_x \rightarrow P \iff \exists x (A_x \rightarrow P)
$$

Examples:

a. If some/any student passes, Ron will (no cross-reference).

$$
\exists x (\text{Student}(x) \land \text{Pass}(x)) \rightarrow \text{Pass}(\text{Ron}) \iff \forall x [(\text{Student}(x) \land \text{Pass}(x)) \rightarrow \text{Pass}(\text{Ron})]
$$

the scope of $\exists x$ the scope of $\forall x$

b. If every student passes, then Ron will.

$$
\forall x(\text{Student}(x) \rightarrow \text{Pass}(x)) \rightarrow \text{Pass}(\text{Ron}) \iff \exists x [(\text{Student}(x) \rightarrow \text{Pass}(x)) \rightarrow \text{Pass}(\text{Ron})]
$$

the scope of $\forall x$ the scope of $\exists x$

Rule 2: For other cases (except biconditionals), simply move quantifiers to change its scope.

$$
P \rightarrow \exists x A_x \iff \exists x (P \rightarrow A_x)
$$

$$
P \rightarrow \forall x A_x \iff \forall x (P \rightarrow A_x)
$$

$$
\exists x A_x \lor P \iff \exists x (A_x \lor P)
$$

$$
\forall x A_x \lor P \iff \forall x (A_x \lor P)
$$

$$
\exists x A_x \land P \iff \exists x (A_x \land P)
$$

$$
\forall x A_x \land P \iff \forall x (A_x \land P)
$$

Examples:

a. $\text{Small}(a) \rightarrow \forall x (\text{(Cube}(x) \land x \neq a) \rightarrow \text{Large}(x))$

$$
\iff \forall x [\text{Small}(a) \rightarrow (\text{Cube}(x) \land x \neq a) \rightarrow \text{Large}(x)]
$$

b. $\exists x \text{Cube}(x) \rightarrow \forall x (\neg \text{Cube}(x) \rightarrow \text{Large}(x))$

$$
\iff \exists y \text{Cube}(y) \rightarrow \forall x (\neg \text{Cube}(x) \rightarrow \text{Large}(x))
$$

$$
\iff \forall x [\exists y \text{Cube}(y) \rightarrow (\neg \text{Cube}(x) \rightarrow \text{Large}(x))]
$$

$$
\iff \forall x \forall y [\text{Cube}(x) \rightarrow (\neg \text{Cube}(y) \rightarrow \text{Large}(y))]
$$
or

$$
\iff \exists x \text{Cube}(x) \rightarrow \forall y (\neg \text{Cube}(y) \rightarrow \text{Large}(y))
$$

$$
\iff \forall x [\text{Cube}(x) \rightarrow \forall y (\neg \text{Cube}(y) \rightarrow \text{Large}(y))]
$$

$$
\iff \forall x \forall y [\text{Cube}(x) \rightarrow (\neg \text{Cube}(y) \rightarrow \text{Large}(y))]
$$
c. \[ \text{Small}(b) \land \forall x \ (\text{Cube}(x) \rightarrow \text{Small}(x)) \]
\[ \iff \forall x [\text{Small}(b) \land (\text{Cube}(x) \rightarrow \text{Small}(x))] \]

d. \[ \exists x \ \text{Cube}(x) \lor \forall x \ \text{Tet}(x) \]
\[ \iff \exists x \ \text{Cube}(x) \lor \forall y \ \text{Tet}(y) \]
\[ \iff \exists x [\forall y (\text{Cube}(x) \lor \text{Tet}(y))] \]
\[ \iff \exists x \forall y (\text{Cube}(x) \lor \text{Tet}(y)) \]

Question: which one is better between two logical equivalent sentences with different scopes, such as, “\( \text{Small}(a) \rightarrow \forall x ( (\text{Cube}(x) \land x \neq a) \rightarrow \text{Large}(x)) \)” and “\( \forall x [\text{Small}(a) \rightarrow (\text{Cube}(x) \land x \neq a) \rightarrow \text{Large}(x)] \)”?

Tips:
(1) Put x-quantifier in front of the formula containing x as closely as possible.
(2) The sentence which quantifiers have smaller scopes is better.
Therefore, "\( P \rightarrow \exists x \ \forall x \)" is better than "\( \exists x (P \rightarrow \forall x) \)"

5. The merge of two quantifiers

\[ \forall x \ P(x) \land \forall x \ Q(x) \iff \forall x (P(x) \land Q(x)) \]
\[ \exists x \ P(x) \lor \exists x \ Q(x) \iff \exists x (P(x) \lor Q(x)) \]

Examples:
\[ \forall x \ \text{Cube}(x) \land \forall x \ \text{Small}(x) \iff \forall x (\text{Cube}(x) \land \text{Small}(x)) \quad \text{All blocks are small cubes.} \]
\[ \exists x \ \text{Cube}(x) \lor \exists x \ \text{Tet}(x) \iff \exists x (\text{Cube}(x) \lor \text{Tet}(x)) \quad \text{Some block is either cube or tetrahedron.} \]

However,
\[ \forall x \ P(x) \lor \forall x \ Q(x) \iff \forall x (P(x) \lor Q(x)) \quad \text{WRONG!!!} \]
\[ \exists x \ P(x) \land \exists x \ Q(x) \iff \exists x (P(x) \land Q(x)) \quad \text{WRONG!!!} \]

For example:
\[ \forall x \ \text{Cube}(x) \lor \forall x \ \text{Tet}(x) \iff \forall x (\text{Cube}(x) \lor \text{Tet}(x)) \quad \text{WRONG!!!} \]

“Either all blocks are cube or all blocks are tetrahedra.”

\[ \exists x \ \text{Cube}(x) \land \exists x \ \text{Tet}(x) \iff \exists x (\text{Cube}(x) \land \text{Tet}(x)) \quad \text{WRONG!!!} \]

“Some block is cube and some block is tetrahedron.”

\[ \exists x (\text{Cube}(x) \land \text{Tet}(x)) \iff \exists x \ (\text{Cube}(x) \land \text{Tet}(x)) \quad \text{WRONG!!!} \]

“Some block is both cube and tetrahedron.”
6. Class practice
(1) Prove the following:
   - A-sentence: \( \forall x (S(x) \rightarrow P(x)) \iff \neg \exists x (S(x) \land \neg P(x)) \)
     the negation of the corresponding O-sentence
   - E-sentence: \( \forall x (S(x) \rightarrow \neg P(x)) \iff \neg \exists x (S(x) \land P(x)) \)
     the negation of the corresponding I-sentence
   - I-sentence: \( \exists x (S(x) \land P(x)) \iff \neg \forall x (S(x) \rightarrow \neg P(x)) \)
     the negation of the corresponding E-sentence
   - O-sentence: \( \exists x (S(x) \land \neg P(x)) \iff \neg \forall x (S(x) \rightarrow P(x)) \)
     the negation of the corresponding A-sentence

(2) Prove that \( \exists x \, Ax \rightarrow P \iff \forall x (Ax \rightarrow P) \) and \( \forall x \, Ax \rightarrow P \iff \exists x (Ax \rightarrow P) \)

Proof:
\[
\exists x \, Ax \rightarrow P \\
\iff \neg \exists x \, Ax \lor P \\
\iff \forall x \, \neg Ax \lor P \\
\iff \forall x \, (\neg Ax \lor P) \\
\iff \forall x \, (Ax \rightarrow P)
\]

(3) Prove that \( P \rightarrow \exists x \, Ax \iff \exists x (P \rightarrow Ax) \) and \( P \rightarrow \forall x \, Ax \iff \forall x (P \rightarrow Ax) \)

Proof:
\[
P \rightarrow \exists x \, Ax \\
\iff \neg P \lor \exists x \, Ax \\
\iff \exists x (\neg P \lor Ax) \\
\iff \exists x (P \rightarrow Ax)
\]

Proof:
\[
P \rightarrow \forall x \, Ax \\
\iff \neg P \lor \forall x \, Ax \\
\iff \forall x (\neg P \lor Ax) \\
\iff \forall x (P \rightarrow Ax)
\]
(4) Prove that
\[ \neg \exists x \ [\text{Cube}(x) \land \forall y \ (\text{Tet}(y) \rightarrow \text{FrontOf}(y, x))] \iff \forall x \exists y \ [\text{Cube}(x) \rightarrow (\text{Tet}(y) \land \neg \text{FrontOf}(y, x))] \]

Proof:
\[ \neg \exists x \ [\text{Cube}(x) \land \forall y \ (\text{Tet}(y) \rightarrow \text{FrontOf}(y, x))] \]
\[ \iff \forall x \neg [\text{Cube}(x) \land \forall y \ (\text{Tet}(y) \rightarrow \text{FrontOf}(y, x))] \]
\[ \iff \forall x [\neg \text{Cube}(x) \lor \neg \forall y \ (\text{Tet}(y) \rightarrow \text{FrontOf}(y, x))] \]
\[ \iff \forall x [\neg \text{Cube}(x) \lor \exists y \ (\neg \text{Tet}(y) \rightarrow \text{FrontOf}(y, x))] \]
\[ \iff \forall x [\neg \text{Cube}(x) \lor \exists y \ (\text{Tet}(y) \land \neg \text{FrontOf}(y, x))] \]
\[ \iff \forall x \exists y [\neg \text{Cube}(x) \land (\text{Tet}(y) \land \neg \text{FrontOf}(y, x))] \]

(5) Prove that
\[ \neg \exists x (\text{Cube}(x) \land \neg \text{Small}(x)) \lor \forall x (\text{Tet}(x) \rightarrow \text{Large}(x)) \iff \forall y \exists x [(\neg \text{Cube}(y) \lor \text{Small}(y)) \land \text{Tet}(x) \land \neg \text{Large}(x)] \]

proof:
\[ \neg \exists x (\text{Cube}(x) \land \neg \text{Small}(x)) \lor \forall x (\text{Tet}(x) \rightarrow \text{Large}(x)) \]
\[ \iff \neg \exists x (\text{Cube}(x) \land \neg \text{Small}(x)) \land \neg \forall x (\text{Tet}(x) \rightarrow \text{Large}(x)) \]
\[ \iff \forall x [\neg \text{Cube}(x) \lor \neg \text{Small}(x)] \land \exists x \ (\neg \text{Tet}(x) \rightarrow \text{Large}(x)) \]
\[ \iff \forall x [\neg \text{Cube}(x) \lor \text{Small}(x)] \land \exists x \ (\text{Tet}(x) \land \neg \text{Large}(x)) \]
\[ \iff \forall x \exists y [(\neg \text{Cube}(y) \lor \text{Small}(y)) \land \text{Tet}(x) \land \neg \text{Large}(x)] \]

(6) Prove that
\[ \forall x [((\text{Cube}(x) \land \exists y (\text{Tet}(y) \land \text{LeftOf}(x, y))) \rightarrow \exists y (\text{Dodec}(y) \land \text{RightOf}(x, y))] \]
\[ \iff \forall x \forall y \exists z [(\text{Cube}(x) \land \text{Tet}(y) \land \text{LeftOf}(x, y)) \rightarrow (\text{Dodec}(z) \land \text{RightOf}(x, z))] \]

Proof:
\[ \forall x [((\text{Cube}(x) \land \exists y (\text{Tet}(y) \land \text{LeftOf}(x, y))) \rightarrow \exists y (\text{Dodec}(y) \land \text{RightOf}(x, y))] \]
\[ \iff \forall x [\exists y (\text{Cube}(x) \land \text{Tet}(y) \land \text{LeftOf}(x, y)) \rightarrow \exists y (\text{Dodec}(y) \land \text{RightOf}(x, y))] \]
\[ \iff \forall x [\exists y (\text{Cube}(x) \land \text{Tet}(y) \land \text{LeftOf}(x, y)) \rightarrow \exists z (\text{Dodec}(z) \land \text{RightOf}(x, z))] \]
\[ \iff \forall x \forall y \exists z [(\text{Cube}(x) \land \text{Tet}(y) \land \text{LeftOf}(x, y)) \rightarrow (\text{Dodec}(z) \land \text{RightOf}(x, z))] \]

(7) Prove that
\[ \exists x \text{ Cube}(x) \rightarrow \exists x \text{ Small}(x) \iff \exists y \forall x (\neg \text{Cube}(x) \lor \text{Small}(y)) \]

Proof:
\[ \exists x \text{ Cube}(x) \rightarrow \exists x \text{ Small}(x) \]
\[ \iff \exists x \text{ Cube}(x) \rightarrow \exists y \text{ Small}(y) \]
\[ \iff \exists y (\exists x \text{ Cube}(x) \rightarrow \text{Small}(y)) \]
\[ \iff \exists y (\neg \exists x \text{ Cube}(x) \lor \text{Small}(y)) \]
\[ \iff \exists y \forall x (\neg \text{Cube}(x) \lor \text{Small}(y)) \]
(8) Prove that $\exists x \left[ \exists y (\text{Cube}(x) \land \text{Tet}(y)) \land \forall y (\text{Dodec}(y) \rightarrow \text{LeftOf}(y, x)) \right]$

$\iff \exists x \exists y \forall z \left[ \text{Cube}(x) \land \text{Tet}(y) \land (\text{Dodec}(z) \rightarrow \text{LeftOf}(z, x)) \right]$

Proof:

$\exists x \left[ \exists y (\text{Cube}(x) \land \text{Tet}(y)) \land \forall y (\text{Dodec}(y) \rightarrow \text{LeftOf}(y, x)) \right]$

$\iff \exists x \exists y \left[ \text{Cube}(x) \land \text{Tet}(y) \land (\forall z (\text{Dodec}(z) \rightarrow \text{LeftOf}(z, x))) \right]$

$\iff \exists x \exists y \forall z \left[ \text{Cube}(x) \land \text{Tet}(y) \land (\text{Dodec}(z) \rightarrow \text{LeftOf}(z, x)) \right]$

1.5.5. Formal Proofs Involving Quantifiers (5.11 of the Text)

1. Derivation rules for quantifiers

- Universal quantifier rules

```
\[ \forall x \ S(x) \]
\[ : \]
\[ : \]
\[ \Diamond S(c) \quad \forall\text{Elim: } \#m \]
```

“c” is any term which you need.

- Existential quantifier rules

```
\[ \exists x \ S(x) \]
\[ : \]
\[ : \]
\[ \exists x \ S(x) \quad \exists\text{Intro: } \#m \]
```

“c” stands for any individual constant.

2. Tips of proofs

- If the conclusion is a universal sentence, try $\forall\text{Intro}$ rule.
- $\forall\text{Elim}$ rule can help you get rid of universal quantifier $\forall x$ (so you can use other rules leant before).
- If both the premises and the conclusion contain existential sentences (and if the conclusion is an existential sentence), then try $\exists\text{Elim}$ rule.
- $\exists\text{Intro}$ rule can help you get rid of unwanted term used in the assumption of $\exists\text{Elim}$ rule.
3. Class exercises

Example 1: \{∀x (Cube(x) → Large(x)), ∀x Cube(x)\} \vdash ∀x Large(x)

1. ∀x (Cube(x) → Large(x))
2. ∀x Cube(x)
   (a)
   3. Cube(a)       ∀Elim: 2
   4. Cube(a) → Large(a)    ∀Elim: 1
   5. Large(a)       →Elim: 3, 4
   6. ∀x Large(x)     ∀Intro: (a)-5.

Example 2: \{∀x (Tet(x) → Small(x)), ∀x (Small(x) → RightOf(x, a))\} \vdash ∀x (Tet(x) → RightOf(x, a))

1. ∀x (Tet(x) → Small(x))
2. ∀x (Small(x) → RightOf(x, a))
   (e)
   3. Tet(e) → Small(e)       ∀Elim: 1
   4. Small(e) → RightOf(e, a)    ∀Elim: 2
      5. Tet(e)
      6. Small(e)       →Elim: 5, 3
      7. RightOf(e, a)       →Elim: 6, 4
      8. Tet(e) → RightOf(e, a)    →Intro: 5-7
   9. ∀x (Tet(x) → RightOf(x, a))     ∀Intro: (e)-8.

Example 3: \{∀x (Tet(x) → Small(x)), ∀x (Small(x) → RightOf(x, a)), ∃x Tet(x)\} \vdash ∃x (Small(x) ∧ RightOf(x, a))

1. ∀x (Tet(x) → Small(x))
2. ∀x (Small(x) → RightOf(x, a))
3. ∃x Tet(x)
   (b)
   4. Tet(b)
   5. Tet(b) → Small(b)       ∀Elim: 1
   6. Small(b) → RightOf(b, a)    ∀Elim: 2
   7. Small(b)       →Elim: 4, 5
   8. RightOf(b, a)       →Elim: 7, 6
   9. Small(b) ∧ RightOf(b, a)       ∧ Intro: 7, 8
      10. ∃x (Small(x) ∧ RightOf(x, a))     ∃Intro: 9
   11. ∃x (Small(x) ∧ RightOf(x, a))     ∃Elim: 3, (b)-10
Example 4: \( \{ \neg \forall x \ P(x) \} \models \exists x \ P(x) \)

1. \( \neg \forall x \ P(x) \)
2. \( \neg \exists x \ P(x) \)
   3. \( \neg P(a) \)
   4. \( \exists x \neg P(x) \)
   5. \( \neg \exists x \ P(x) \land \exists x \ P(x) \)
      \( \land \) Intro: 2, 4
6. \( \neg P(a) \)
7. \( P(a) \)
   8. \( \forall x \ P(x) \)
      \( \forall \) Intro: (a)-7
9. \( \forall x \ P(x) \land \neg \forall x \ P(x) \)
     \( \land \) Intro: 8, 1
10. \( \neg \exists x \neg P(x) \)
      \( \neg \) Intro: 2-9
11. \( \exists x \neg P(x) \)
      \( \neg \) Elim: 10.

Example 5: \( \{ \forall x \ (\text{Small}(x) \lor \text{Large}(x)), \forall x \ (\text{Small}(x) \rightarrow (\text{Dodec}(x) \lor \text{Tet}(x))), \exists x \neg \text{Dodec}(x) \} \models \exists x \ (\text{Large}(x) \lor \text{Tet}(x)) \)

1. \( \forall x \ (\text{Small}(x) \lor \text{Large}(x)) \)
2. \( \forall x \ (\text{Small}(x) \rightarrow (\text{Dodec}(x) \lor \text{Tet}(x))) \)
3. \( \exists x \neg \text{Dodec}(x) \)
   4. \( \neg \text{Dodec}(a) \)
      \( \neg \) Intro: 3
6. \( \text{Small}(a) \lor \text{Large}(a) \)
   7. \( \forall x \ (\text{Small}(x) \rightarrow (\text{Dodec}(x) \lor \text{Tet}(x))) \)
      \( \forall \) Elim: 1
8. \( \neg \text{Tet}(a) \)
   9. \( \text{Tet}(a) \)
10. \( \text{Tet}(a) \)
11. \( \neg \text{Dodec}(a) \)
12. \( \neg \text{Tet}(a) \)
13. \( \text{Dodec}(a) \land \neg \text{Dodec}(a) \)
      \( \land \) Intro: 11, 4
14. \( \neg \neg \neg \text{Dodec}(a) \land \neg \text{Dodec}(a) \)
      \( \neg \neg \neg \) Intro: 12-13
15. \( \neg \text{Tet}(a) \)
16. \( \neg \text{Tet}(a) \)
17. \( \text{Tet}(a) \lor \text{Tet}(a) \)
      \( \lor \) Elim: 8, 9-10, 11-15
18. \( \text{Large}(a) \)
19. \( \text{Large}(a) \lor \text{Tet}(a) \)
      \( \lor \) Intro: 18
20. \( \text{Large}(a) \lor \text{Tet}(a) \)
      \( \lor \) Elim: 5, 6-17, 18-19
21. \( \exists x \ (\text{Large}(x) \lor \text{Tet}(x)) \)
      \( \exists \) Intro: 20
22. \( \exists x \ (\text{Large}(x) \lor \text{Tet}(x)) \)
      \( \exists \) Elim: 3, (a)-21
Example 6: \( \{ \} \vdash \forall x (P(x) \rightarrow Q(x)) \rightarrow (\forall x P(x) \rightarrow \forall x Q(x)) \)

1. \( \forall x (P(x) \rightarrow Q(x)) \)
2. \( \forall x P(x) \)
   \[ \begin{array}{l}
   3. P(b) \\
   4. P(b) \rightarrow Q(b) \\
   5. Q(b) \\
   6. \forall x Q(x) \\
   7. \forall x (P(x) \rightarrow \forall x Q(x)) \\
   8. \forall (P(x) \rightarrow Q(x)) \rightarrow (\forall x P(x) \rightarrow \forall x Q(x))
   \end{array} \]

\( \forall \text{ Elim: 2} \)
\( \forall \text{ Elim: 1} \)
\( \rightarrow \text{ Elim: 5, 4} \)
\( \forall \text{ Intro: (b)-5} \)
\( \rightarrow \text{ Intro: 2-6} \)
\( \rightarrow \text{ Intro: 1-7} \)
1.6. Notes for Chapter 6: Multiple Quantifiers

1.6.1. Translations (6.1-6.4 of the Text)

1. Unmixed Quantifiers (same quantity)

1.1 Examples

Some cube is to the left of some tetrahedron.
\[ \exists x \ [\text{Cube}(x) \land \exists y (\text{Tet}(y) \land \text{LeftOf}(x, y))] \]

or
\[ \exists x \exists y [\text{Cube}(x) \land \text{Tet}(y) \land \text{LeftOf}(x, y)] \]

Every cube is to the left of every tetrahedron.
\[ \forall x [\text{Cube}(x) \rightarrow \forall y (\text{Tet}(y) \rightarrow \text{LeftOf}(x, y))] \]

or
\[ \forall x \forall y [\text{Cube}(x) \rightarrow (\text{Tet}(y) \rightarrow \text{LeftOf}(x, y))] \]

or
\[ \forall x \forall y [(\text{Cube}(x) \land \text{Tet}(y)) \rightarrow \text{LeftOf}(x, y)] \]

1.2 How to distinguish two different objects? (The relation between two variables)

Domain: people in JC or blocks in a Tarski’s world

Warning: two distinct variables do not necessarily refer to two distinct objects

\begin{align*}
\text{Jenny loves someone (this person could be Jenny herself).} & \quad \exists x \ \text{Love}(\text{Jenny}, x) \\
\text{Jenny loves someone else (excluding Jenny herself).} & \quad \exists x (x \neq \text{Jenny} \land \text{Love}(\text{Jenny}, x))
\end{align*}

\begin{align*}
\text{Jenny loves everyone.} & \quad \forall x \ \text{Love}(\text{Jenny}, x) \\
\text{Jenny loves everyone else.} & \quad \forall x (x \neq \text{Jenny} \rightarrow \text{Love}(\text{Jenny}, x))
\end{align*}

\begin{align*}
\text{Jenny does not read } \text{Hamlet} \text{ but everyone else in the class does.} & \quad \neg \ \text{Read}(\text{Jenny, Hamlet}) \land \forall x (x \neq \text{Jenny} \rightarrow \text{Read}(x, \text{Hamlet}))
\end{align*}

- Think of a Tarski’s world with four cubes in the same row. You want to say a true sentence that every cube is either to the left or to the right of every (other) cube. Which of the followings is correct translation?

\[ \forall x \forall y [(\text{Cube}(x) \land \text{Cube}(y)) \rightarrow (\text{RightOf}(x, y) \lor \text{LeftOf}(x, y))] \quad \text{a false sentence!} \]

\[ \forall x \forall y [(\text{Cube}(x) \land \text{Cube}(y) \land x \neq y) \rightarrow (\text{RightOf}(x, y) \lor \text{LeftOf}(x, y))] \quad \text{a true sentence!} \]

- Similarly, if you want to say “There are at least two cubes”, the correct translation is:

\[ \exists x \exists y (\text{Cube}(x) \land \text{Cube}(y) \land x \neq y) \]

But NOT \[ \exists x \exists y (\text{Cube}(x) \land \text{Cube}(y)) \]

It is a wrong translation since it means that there is at least one cube.

\[ x \text{ and } y \text{ could refer to the same cube, not necessarily the two distinct cubes.} \]
• However, there are many other ways to distinguish two objects other than using “≠”. For example, we can distinguish two objects by specifying some relation between them. If you want to say that “a is between two cubes”, the following translation will be okay since “Between(a, x, y)” has indicated that x and y refer to different objects. Therefore, here “x ≠ y” is not necessary and is redundant.

\[ \exists x \exists y \ (\text{Cube}(x) \land \text{Cube}(y) \land \text{Between}(a, x, y)) \]

Similarly, If you want to say “There are at least two cubes and one is to the right of the other”, then the following translation will be sufficient (here “x ≠ y” will be redundant):

\[ \exists x \exists y \ (\text{Cube}(x) \land \text{Cube}(y) \land \text{RightOf}(x, y)) \]

Still another way to make the distinction: “There are at least two cubes, one of them is small and another is large.” \( \exists x \exists y \ (\text{Cube}(x) \land \text{Cube}(y) \land \text{Small}(x) \land \text{Large}(y)) \)

2. Mixed Quantifiers

2.1 Examples

Every cube is to the left of a (some) tetrahedron.

\[ \forall x [\text{Cube}(x) \rightarrow \exists y (\text{Tet}(y) \land \text{LeftOf}(x, y))] \]

or \[ \forall x \exists y [\text{Cube}(x) \rightarrow (\text{Tet}(y) \land \text{LeftOf}(x, y))] \]

Some cube is to the left of every tetrahedron.

\[ \exists x [\text{Cube}(x) \land \forall y (\text{Tet}(y) \rightarrow \text{LeftOf}(x, y))] \]

or \[ \exists x \forall y [\text{Cube}(x) \land (\text{Tet}(y) \rightarrow \text{LeftOf}(x, y))] \]

2.2 The order of mixed quantifiers

• Rule: If all quantifiers have the same quantity (all are universals or all are existentials), then the order of quantifiers does not matter.

Someone likes someone. \( \exists x \exists y \ \text{Like}(x, y) \leftrightarrow \exists y \exists x \ \text{Like}(x, y) \)

Everyone likes everyone. \( \forall x \forall y \ \text{Like}(x, y) \leftrightarrow \forall y \forall x \ \text{Like}(x, y) \)

• Rule: If quantifiers are mixed, then the order of quantifiers does matter (had better not to change the order).

Everyone likes someone (may be different persons). \( \forall x \exists y \ \text{Like}(x, y) \)

Someone (the same person) is liked by everyone. \( \exists y \forall x \ \text{Like}(x, y) \)

Every student admires someone who has red hair

\[ \forall x [\text{Student}(x) \rightarrow \exists y (\text{Person}(y) \land \text{RedHair}(y) \land \text{Admire}(x, y))] \]

\[ \leftrightarrow \forall x \exists y [\text{Student}(x) \rightarrow (\text{Person}(y) \land \text{RedHair}(y) \land \text{Admire}(x, y))] \]

Someone who has red hair is admired by every student.

\[ \exists y [\text{Person}(y) \land \text{RedHair}(y) \land \forall x (\text{Student}(x) \rightarrow \text{Admire}(x, y))] \]

\[ \leftrightarrow \exists y \forall x [\text{Person}(y) \land \text{RedHair}(y) \land (\text{Student}(x) \rightarrow \text{Admire}(x, y))] \]
3. More translations

3.1 Multiple quantifiers
Domain: All people in JC

- Someone likes someone.  \( \exists x \exists y \text{Like} (x, y) \)
- Everyone likes everyone.  \( \forall x \forall y \text{Like} (x, y) \)
- Everyone likes someone (different persons).  \( \forall x \exists y \text{Like} (x, y) \)
- Someone (the same person) likes everyone.  \( \exists x \forall y \text{Like} (x, y) \)
- Everyone is liked by someone (different persons).  \( \forall x \exists y \text{Like} (y, x) \)
- Someone (the same person) is liked by everyone.  \( \exists x \forall y \text{Like} (y, x) \)

Some student likes some student.  \( \exists x \exists y [\text{Student} (x) \land \text{Student} (y) \land \text{Like}(x, y)] \)
Every student likes every student.  \( \forall x \forall y [(\text{Student} (x) \land \text{Student} (y)) \rightarrow \text{Like}(x, y)] \)
Every student likes some student.  \( \forall x [\text{Student} (x) \rightarrow \exists y (\text{Student} (y) \land \text{Like} (x, y))] \)
Some student likes every student.  \( \exists x [\text{Student} (x) \land \forall y (\text{Student} (y) \rightarrow \text{Like} (x, y))] \)
Every student is liked by some student.  \( \forall x [\text{Student} (x) \rightarrow \exists y (\text{Student} (y) \land \text{Like} (y, x))] \)
Some student is liked by every student.  \( \exists x [\text{Student} (x) \land \forall y (\text{Student} (y) \rightarrow \text{Like} (y, x))] \)

3.2 Exceptional Sentences
Domain: All students in logic class.

- What is excluded is a singular object

  \( \text{All } S \text{ but (Every } S \text{ but / except) } a \text{ are } P. \quad \sim P(a) \land \forall x \ [x \neq a \land S(x) \rightarrow P(x)] \)

  \( \text{All cubes but } a \text{ are small.} \quad \sim \text{Small}(a) \land \text{Cube}(a) \land \forall x [(x \neq a \land \text{Cube}(x)) \rightarrow \text{Small}(x)] \)

  \( \text{Every student but } Jenny \text{ passed the logic test.} \quad \sim \text{Pass}(Jenny) \land \forall x (x \neq Jenny \rightarrow \text{Pass}(x)) \)

  \( \text{Some } S \text{ but } a \text{ is } P. \quad \sim P(a) \land \forall x \ [x \neq a \land S(x) \land P(x)] \)

  \( \text{Some cube but } a \text{ is small.} \quad \sim \text{Small}(a) \land \text{Cube}(a) \land \exists x [\text{Cube}(x) \land x \neq a \land \text{Small}(x)] \)

  \( \text{Someone but } Jenny \text{ gets an A in the test.} \quad \sim \text{GetA}(Jenny) \land \exists x [x \neq Jenny \land \text{GetA}(x)] \)

Comparison:

\[
\begin{align*}
\text{Only } a \text{ and } b \text{ are not cube.} & \quad \sim [\text{Cube}(a) \lor \text{Cube}(b)] \land \forall x [(x \neq a \lor x \neq b) \rightarrow \text{Cube}(x)] \\
\text{Only } a \text{ and } b \text{ are cube.} & \quad [\text{Cube}(a) \land \text{Cube}(b)] \land \forall x [\text{Cube}(x) \rightarrow (x = a \lor x = b)]
\end{align*}
\]
• What are excluded are a group of objects

**All but (Everyone but) S are P.**
\[ ∀x (¬S(x) → P(x)) \wedge ∀x (S(x) → ¬P(x)) \]
\[ ⇔ ∀x [(¬S(x) → P(x)) \wedge (S(x) → ¬P(x))]) \]
\[ ⇔ ∀x (∼S(x) ↔ P(x)) \]
\[ ⇔ ∀x (S(x) ↔ ¬P(x)) \]

**All blocks but cubes are small.**
\[ ∀x (Cube(x) → ∼Small(x)) \wedge ∀x (∼Cube(x) → Small(x)) \]

**All students but girls attend the class.**
\[ ∀x (∼Girl(x) → Attend(x)) \wedge ∀x (Girl(x) → ∼Attend(x)) \]

**All P but S are Q.**
\[ ∀x [ (P(x) ∧ ∼S(x)) → Q(x)] \wedge ∀x [ (P(x) ∧ S(x)) → ∼Q(x)] \]
All cubes but those in back of a are small.
\[ ∀x [(Cube(x) ∧ ∼BackOf(x, a)) → Small(x)] \wedge ∀x [(Cube(x) ∧ BackOf(x, a)) → ∼Small(x)] \]

**Some P but S are Q.**
\[ ∃x [ (P(x) ∧ ∼S(x)) ∧ Q(x)] \wedge ∀x [ (P(x) ∧ S(x)) → ∼Q(x)] \]
Some cubes but those in back of a are small.
\[ ∃x [(Cube(x) ∧ ∼BackOf(x, a)) ∧ Small(x)] \wedge ∀x [(Cube(x) ∧ BackOf(x, a)) → ∼Small(x)] \]

**Some but S are P.**
\[ ∃x (∼S(x) ∧ P(x)) \wedge ∀x (S(x) → ∼P(x)) \]
Some blocks but cubes are small.
\[ ∃x (∼Cube(x) ∧ Small(x)] \wedge ∀x (Cube(x) → ∼Small(x)) \]
Some students but those who miss the review session got As in the test.
\[ ∃x (∼Miss(x) ∧ GetA(x)) \wedge ∀x (miss(x) → ∼GetA(x)) \]

### 3.3 Partial and complete negation

- **Tip:** If it is hard to translate a negative sentence (E or O sentence), then translate it into the negation of a corresponding affirmative sentence (A or I sentence).

<table>
<thead>
<tr>
<th>E-sentence</th>
<th>the negation of I-sentence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nobody loves anyone /someone</td>
<td>⇔ ∼(Someone loves someone) \⇔ ∀x∃y Like(x, y)</td>
</tr>
<tr>
<td>Nobody loves everyone.</td>
<td>⇔ ∼(Someone loves everyone). \⇔ ∀x∀y Like(x, y)</td>
</tr>
<tr>
<td>Nobody loves no one.</td>
<td>⇔ ∼(Someone loves no one) \⇔ ∀y ∼Like(x, y)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>O-sentence</th>
<th>the negation of A-sentence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Someone does not love anyone/someone.</td>
<td>⇔ ∼(Everyone loves someone) \⇔ ∃y ∀x ∼Like(x, y)</td>
</tr>
<tr>
<td>Someone does not love everyone.</td>
<td>⇔ ∼(Everyone loves everyone) \⇔ ∃y ∀x ∼Like(x, y)</td>
</tr>
<tr>
<td>Not all people love someone.</td>
<td>⇔ ∼(All people love someone) \⇔ ∃y ∀x ∼Like(x, y)</td>
</tr>
<tr>
<td>Not all people love everyone.</td>
<td>⇔ ∼(All people love everyone) \⇔ ∃y ∀x ∼Like(x, y)</td>
</tr>
</tbody>
</table>
1.7. Notes for Chapter 7: Some Specific Uses of Quantifiers

1.7.1. Numerical Claims and Definite Descriptions (7.1-7.2 of the Text)

7.1 Numerical Claims

DD: All people in JC or all blocks in a Tarski’s world.

1. At least N

At least one S is P \[\exists x (S(x) \land P(x))\]
At least one student gets an A. \[\exists x (\text{Student}(x) \land \text{GetA}(x))\]
At least one cube is small. \[\exists x (\text{Cube}(x) \land \text{Small}(x))\]
There is at least one cube in front of all tetrahedrons. \[\exists x [\text{Cube}(x) \land \forall y (\text{Tet}(y) \rightarrow \text{FrontOf}(x, y))]\]

At least two S are P \[\exists x \exists y (S(x) \land P(x) \land S(y) \land P(y) \land x \neq y)\]
At least two students fail the final. \[\exists x \exists y [\text{Student}(x) \land \text{Fail}(x) \land \text{Student}(y) \land \text{Fail}(y) \land x \neq y]\]
There are at least two small cubes. \[\exists x \exists y [\text{Cube}(x) \land \text{Cube}(y) \land \text{Small}(x) \land \text{Small}(y) \land x \neq y]\]

At least three S are P \[\exists x \exists y \exists z [S(x) \land S(y) \land S(z) \land P(x) \land P(y) \land P(z) \land x \neq y \land x \neq z \land y \neq z]\]
There are at least three students. \[\exists x \exists y \exists z [\text{Student}(x) \land \text{Student}(y) \land \text{Student}(z) \land x \neq y \land x \neq z \land y \neq z]\]

2. At most N

At most one S is P \[\sim (\text{at least two S are P})\]
\[\sim \exists x \exists y (S(x) \land P(x) \land S(y) \land P(y) \land x \neq y)\]
\[\iff \forall x \forall y [(S(x) \land P(x) \land S(y) \land P(y)) \rightarrow x = y) \text{ Recommend!}\]

At most one student fails the course.
\[\sim \exists x \exists y [\text{Student}(x) \land \text{Fail}(x) \land \text{Student}(y) \land \text{Fail}(y) \land x \neq y]\]
\[\iff \forall x \forall y [(\text{Student}(x) \land \text{Fail}(x) \land \text{Student}(y) \land \text{Fail}(y)) \rightarrow x = y]\]

At most two S are P \[\sim (\text{at least three S are P})\]
\[\sim \exists x \exists y \exists z [(S(x) \land P(x) \land S(y) \land P(y) \land S(z) \land P(z)) \land x \neq y \land x \neq z \land y \neq z]\]
\[\iff \forall x \forall y \forall z [(S(x) \land P(x) \land S(y) \land P(y) \land S(z) \land P(z)) \rightarrow (x = y \lor x = z \lor y = z)]\]

At most two cubes are large.
\[\sim \exists x \exists y \exists z [(\text{Cube}(x) \land \text{Cube}(y) \land \text{Cube}(z) \land \text{Large}(x) \land \text{Large}(y) \land \text{Large}(z)) \land x \neq y \land x \neq z \land y \neq z]\]
\[\iff \forall x \forall y \forall z [(\text{Cube}(x) \land \text{Cube}(y) \land \text{Cube}(z) \land \text{Large}(x) \land \text{Large}(y) \land \text{Large}(z)) \rightarrow (x = y \lor x = z \lor y = z)]\]
3. Exact N

Formula: exact N = at least N and at most N

3.1 Exact one / Only one / One and only one / None but one

3.1.1 When the subject is a general term "S"

(1) There is exactly one P

\[ \exists x \, P(x) \land \neg \exists y \, (P(y) \land x \neq y) \]

at least one  at most one (not at least two)

\[ \iff \exists x \, [P(x) \land \neg \exists y \, (P(y) \land x \neq y)] \]

at least one

\[ \iff \exists x \, [P(x) \land \forall y \, (P(y) \rightarrow y = x)] \quad \text{Recommend!} \]

\[ \iff \exists x \forall y (P(y) \leftrightarrow y = x) \]

There is exactly/only one cube.  \( \exists \, [\text{Cube}(x) \land \forall y \, \text{Cube}(y) \rightarrow y = x] \)

There is one and only one block.  \( \exists [\text{Cube}(x) \lor \text{Tet}(x) \lor \text{Dodec}(x)] \land \forall y \, ((\text{Cube}(y) \lor \text{Tet}(y) \lor \text{Dodec}(y)) \rightarrow y = x] \)

(2) Exactly/only one S is P.  \( \iff \) At least one S is P AND at most one S is P

\[ \exists x \, (S(x) \land P(x)) \land \neg \exists y \, (S(y) \land P(y) \land x \neq y) \]

at least one  at most one (not at least two)

\[ \iff \exists x \, [ (S(x) \land P(x)) \land \neg \exists y \, ((S(y) \land P(y)) \land x \neq y) \]

at least one

\[ \iff \exists x \, [ (S(x) \land P(x)) \land \forall y \, ((S(y) \land P(y)) \rightarrow y = x)] \quad \text{recommend!} \]

\[ \iff \exists x \forall y \, ((S(y) \land P(y)) \leftrightarrow y = x) \]

Exactly/Only one student gets an A.

\( \exists x [(\text{Student}(x) \land \text{GetA}(x)) \land \forall y ((\text{Student}(y) \land \text{GetA}(y)) \rightarrow x = y)] \)

Only one person who is rich is taller than Alice.

\( \exists x \, [(\text{Rich}(x) \land \text{Taller}(x, \text{Alice})) \land \forall y \, ((\text{Rich}(y) \land \text{Taller}(y, \text{Alice})) \rightarrow x = y)] \)

Only one cube is small.

\( \exists x \, [(\text{Cube}(x) \land \text{Small}(x)) \land \forall y \, ((\text{Cube}(y) \land \text{Small}(y)) \rightarrow x = y)] \)
Exactly one block is to the right of some block.
\[ \exists x \left[ \exists y \text{RightOf}(x, y) \land \forall z \left( \exists y \text{RightOf}(z, y) \rightarrow z = x \right) \right] \]

(3) The only (one) \( S^* \) is \( P \) ⇔ There is one and only one \( S \) and \( S \) is \( P \).

\[ \exists x \left[ S(x) \land \forall y (S(y) \rightarrow y = x) \land P(x) \right] \]

* \( S \) is singular use of a general term.

The only (one) female student in Juniata College gets an A.
\[ \exists x \left[ \left( \text{Student}(x) \land \text{Female}(x) \right) \land \forall y \left( \left( \text{Student}(y) \land \text{Female}(y) \right) \rightarrow y = x \right) \land \text{GetA}(x) \right] \]

The one and only one rich person is taller than Bill.
\[ \exists x \left[ \text{Rich}(x) \land \forall y \left( \text{Rich}(y) \rightarrow y = x \right) \land \text{Taller}(x, \text{Bill}) \right] \]

The only (one) philosopher on campus is the only (one) socialist.
\[ \exists x \left[ \text{Phil}(x) \land \text{Socialist}(x) \land \forall y \left( \text{Phil}(y) \rightarrow y = x \right) \land \forall z \left( \text{Socialist}(z) \rightarrow z = x \right) \right] \]

Pat is the only child of Jenny's. \( \text{Pat} = \text{Child(Jenny)} \land \forall x \left[ x = \text{Child(Jenny)} \rightarrow x = \text{Pat} \right] \)

Comparison:

\[ \begin{cases} 
\text{Exactly/only one cube is to right of } a \text{ (There might be more than one cube. But only one of them is to the right of } a) & \exists x \left[ \text{Cube}(x) \land \text{RightOf}(x, a) \land \forall y \left( \left( \text{Cube}(y) \land \text{RightOf}(y, a) \right) \rightarrow y = x \right) \right] \\
\text{The only (one) cube is to the right of } a \text{ (There is one and only one cube AND it is to the right of } a). & \exists x \left[ \text{Cube}(x) \land \forall y \left( \text{Cube}(y) \rightarrow y = x \right) \land \text{RightOf}(x, a) \right] \\
\text{Only cubes are to the right of } a \text{ (There might be more than one cube, and there might be other shape of blocks. But only cubes, maybe more than one, can (not necessarily) be to the right of } a). & \forall x \left( \text{RightOf}(x, a) \rightarrow \text{Cube}(x) \right) \\
\text{The only cubes are to the right of } a \text{ (There are cubes and maybe other shapes of blocks. And all cubes, maybe more than one, are to the right of } a). & \forall x \left( \text{Cube}(x) \rightarrow \text{RightOf}(x, a) \right) \\
\text{Only one rich man is mean.} & \exists x \left[ \text{Rich}(x) \land \text{Mean}(x) \land \forall y \left( \left( \text{Rich}(y) \land \text{Mean}(y) \right) \rightarrow y = x \right) \right] \\
\text{The only rich man is mean.} & \exists x \left[ \text{Rich}(x) \land \forall y \left( \text{Rich}(y) \rightarrow y = x \right) \land \text{Mean}(x) \right] 
\end{cases} \]

(4) Other cases

There is one tallest person in the world. \( \exists x \forall y (x \neq y \rightarrow \text{Taller}(x, y)) \)

Every boy has exactly one girl friend. \( \forall x \left[ \text{Boy}(x) \rightarrow \exists y \left( y = \text{girl}(x) \land \forall z \left( \text{girl}(x) \rightarrow z = y \right) \right) \right] \)
3.1.2 When the subject is a singular term "a"

Only (None but) a is P. \[ P(a) \land \forall x \ (P(x) \rightarrow x = a) \]

Only Michael gets an A. \[ \text{GetA (Michael)} \land \forall x \ (\text{GetA}(x) \rightarrow x = \text{Michael}) \]

Only a is a cube. \[ \text{Cube}(a) \land \forall x \ (\text{Cube}(x) \rightarrow x = a) \]

3.2 Exact two

3.2.1 There are exact two P.

\[ \exists x \exists y \ [P(x) \land P(y) \land x \neq y \land \forall z \ (P(z) \rightarrow (z = x \lor z = y))] \]

\[ \leftrightarrow \exists x \exists y \ [(x \neq y \land \forall z \ ((S(z) \land P(z)) \leftrightarrow (z = x \lor z = y))] \]

There are only two cubes. \[ \exists x \exists y \ [\text{Cube}(x) \land \text{Cube}(y) \land x \neq y \land \forall z \ (\text{Cube}(z) \rightarrow (z = x \lor z = y))] \]

3.2.2 Exact two S are P \( \leftrightarrow \) (At least two S are P AND at most two S are P)

\[ \exists x \exists y \ [S(x) \land P(x) \land S(y) \land P(y) \land x \neq y \land \forall z \ ((S(z) \land P(z)) \rightarrow (z = x \lor z = y))] \]

\[ \leftrightarrow \exists x \exists y \ [(x \neq y \land \forall z \ ((S(z) \land P(z)) \leftrightarrow (z = x \lor z = y))] \]

Exactly two cubes are small.

\[ \exists x \exists y \ [\text{Cube}(x) \land \text{Small}(x) \land \text{Cube}(y) \land \text{Small}(y) \land x \neq y \land \forall z \ ((\text{Cube}(z) \land \text{Small}(z)) \rightarrow (z = x \lor z = y))] \]

Only a and b are cubes. \[ \text{Cube}(a) \land \text{Cube}(b) \land \forall x \ [\text{Cube}(x) \rightarrow (x = a \lor x = b)] \]

3.3 Exact three

There are exactly three P.

\[ \exists x \exists y \exists z \ [P(x) \land P(y) \land P(z) \land x \neq y \land x \neq z \land y \neq z \land \forall w \ (P(w) \rightarrow (w = x \lor w = y \lor w = z))] \]

\[ \leftrightarrow \exists x \exists y \exists z \ [(x \neq y \land x \neq z \land y \neq z \land \forall w \ (P(w) \leftrightarrow (w = x \lor w = y \lor w = z))] \]

There are three cubes.

\[ \exists x \exists y \exists z \ [\text{Cube}(x) \land \text{Cube}(y) \land \text{Cube}(z) \land x \neq y \land x \neq z \land y \neq z \land \forall w \ (\text{Cube}(w) \rightarrow (w = x \lor w = y \lor w = z))] \]

7.2 Definite Description

DD: all people in the world or all blocks in a Tarski’s world.

The P is a Q. \( \leftrightarrow \) There exists exactly one / one and only one P, and it is a Q.

\[ \exists x \ [P(x) \land \forall y \ (P(y) \rightarrow y = x) \land Q(x)] \]

There is exactly one P
The present king of France is bald. \iff \text{There exists one and only one present king of France and that person is bald.}
\begin{align*}
\exists x \ [ \ & \text{Bald}(x) \land \forall y (\text{King}(y) \rightarrow y = x) \land \text{King}(x)] \\
\end{align*}

The inventor of the phonograph is an American.
\begin{align*}
\exists x \ [ \ & \text{Invent}(x, \text{phonograph}) \land \forall y (\text{Invent}(y, \text{phonograph}) \rightarrow y = x) \land \text{American}(x)]
\end{align*}

The cube is between two small tetrahedrons.
\begin{align*}
\exists x \ [ \ & \text{Cube}(x) \land \forall u (\text{Cube}(u) \rightarrow u = x) \land \exists y \exists z (\text{Small}(y) \land \text{Small}(z) \land \text{Tet}(y) \land \\
& \text{Tet}(z) \land y \neq z \land \text{Between}(x, y, z))]
\end{align*}

The daughter of John is a physician.
\begin{align*}
\exists x \ [ \ & x = \text{daughter}(\text{John}) \land \forall y (y = \text{daughter}(\text{John}) \rightarrow y = x) \land \text{Physician}(x)]
\end{align*}

The elephant in my closet is wrinkling my clothes.
\begin{align*}
\exists x \ [ \ & \text{Elephant}(x) \land \text{InCloset}(x) \land \forall y ((\text{Elephant}(y) \land \text{InCloset}(y)) \rightarrow y = x) \land \\
& \text{Wrinkling}(x)]
\end{align*}

Comparison:
\[
\begin{cases}
\text{A cube is small.} & \exists x (\text{Cube}(x) \land \text{Small}(x)) \\
\text{Only cubes are small.} & \forall x (\text{Small}(x) \rightarrow \text{Cube}(x)) \\
\text{The only cubes are small.} & \forall x (\text{Cube}(x) \rightarrow \text{Small}(x)) \\
\text{The only cube is small.} & \exists x [\text{Cube}(x) \land \forall y (\text{Cube}(y) \rightarrow y = x) \land \text{Small}(x)] \\
\textbf{The cube} \text{ is small.} & \iff \text{The only cube is small.} \\
& \exists x [\text{Cube}(x) \land \forall y (\text{Cube}(y) \rightarrow y = x) \land \text{Small}(x)] \\
\text{Only one cube is small.} & \exists x [\text{Cube}(x) \land \text{Small}(x) \land \forall y ((\text{Cube}(y) \land \text{Small}(y)) \\
& \rightarrow y = x)]
\end{cases}
\]