1.3. Notes for Chapter 3: Conjunctions, Disjunctions, and Negations

1.3.1. Introduction to Conjunctions, Disjunctions, and Negations (3.1, 3.2, 3.3, and 3.4 of the Text)

1. Introduction
1.1 Moving from atomic sentences to compound sentences

\[
\begin{align*}
\text{Sentences} & \\
\text{Simple sentences:} & \quad \text{one single predicate} \\
& \quad \text{no truth-functional connectives.} \\
\text{Compound sentences:} & \quad \text{More than one predicate} \\
& \quad \text{Two or more simple sentences connected by some truth-functional connectives.}
\end{align*}
\]

For example:

John is a student and Joe is a teacher.

John or Joe is a student = John is a student or Joe is a student.

If John is a student, then Joe is a teacher.

John is not a student.


Definition: a compound sentence is truth-functional iff the truth-value of the sentence is fully determined by the truth-value of its component simple sentences.

Then the connectives connected component sentences of a truth-functional sentence is truth-functional connectives. They are: conjunction, disjunction, negation, and conditional.

For example,

Truth-functional sentences:

\[
\begin{align*}
\text{John is a student and Joe is a teacher.} & \quad \text{(False)} \\
\text{If John is a student, then Joe is a teacher.} & \quad \text{(False)}
\end{align*}
\]
Non-truth-functional sentences:

John loves Kathy because he kisses her.  (True or False)  

True

I believe that Pat is on the mat (propositional attitude).  (True or False)  

True

2. Syntax of truth-functional connectives

<table>
<thead>
<tr>
<th>In English</th>
<th>In FOL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conjunction and, but, however, although, nevertheless, moreover, in additions, ,</td>
<td>$\land$ / &amp;</td>
</tr>
<tr>
<td>Disjunction or, either...or..., at least one of two..., unless</td>
<td>$\lor$</td>
</tr>
<tr>
<td>Negation not, be hardly, unhappy, impossible, incomplete</td>
<td>$\neg$</td>
</tr>
</tbody>
</table>

Attention: two senses of disjunction

Disjunction

In exclusive sense: exactly one alternative (at least one and at most one alternative)

In inclusive sense: at least one alternative (and could be both)

For example:

Waitress: “You can have ice cream or a cake as desert” (but not both).
Alice and Katy’s father: “John, you can marry either Alice or Kathy” (but not both).
Professor: “Joe, you can take either ethics or human nature course to fulfill your philosophy requirement” (sure you can take both if you like).

In FOL, we define “OR” in inclusive sense only.

3. Semantics of truth-functional connectives

3.1 Truth-table definitions: Suppose that P and Q here represent any sentence (either simple or compound sentences). We can define the truth-value of a compound sentence consisting of P and Q as follows:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>$P \land Q$</th>
<th>$P \lor Q$</th>
<th>$\neg P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
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<td>F</td>
<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>
3.2 How to determine the truth value of a sentence?

(1) If the truth-values of component sentences of a compound sentence are given: From inside to outside! (when you determine the truth value of a sentence based on a Tarski’s World).

- Primary connective

\[(P \land Q) \land (Q \lor R)\]
\[
\begin{array}{ccc}
T & F & F \\
F & F & T \\
\end{array}
\]

\[\neg (P \land Q) \land (R \lor P)\]
\[
\begin{array}{ccc}
T & F & F \\
F & T & T \\
\end{array}
\]

(2) If the truth value of a compound sentence is given: from outside to inside! (when you play the game to see your truth commitments)

\[(P \land Q) \land (Q \lor R), \text{ therefore, } P \text{ is true, } Q \text{ is true, and } R \text{ could be true or false.}\]
\[
\begin{array}{ccc}
T & T & T \\
\end{array}
\]

\[\neg (P \land Q) \land (R \lor P)\]
\[F\]

In this case, there are four possible truth values:

- R is false and P is false, and Q could be true or false. OR
- P is true and Q is true, and R could be either true or false.

4. Correct use of parentheses

\[P \land Q \lor R \text{ (WRONG!)}\]
\[
\{ \begin{array}{c}
(P \land Q) \lor R \\
P \land (Q \lor R) \\
\end{array} \}
\]

Which one???

Conventions:

- “\(\neg\)” always apply to the smallest unit right after it:

\[\neg (R \lor (P \lor Q))\]
\[\neg (R \lor P) \lor Q\]

- R \(\lor\) P \(\lor\) Q  Okay!!!
- P \(\land\) Q \(\land\) R  Okay!!!
1.3.2. Logical Equivalency (3.5 of the Text)

1. Definition:

Two sentences are logically equivalent iff they have same truth value in exactly the same circumstances (under any possible interpretation / in any possible world).

Illustrations:

1.1 In the language of the Tarski’s World (with fixed interpretation)

\[ a \text{ is to one side or other of cube } b, \text{ but is in front of dodecahedron } c. \]

Suppose that you have two different translations of the above English sentence as follows:

(a) \[ \text{Cube(b)} \land (\text{LeftOf}(a, b) \lor \text{RightOf}(a, b)) \land \text{Dodec(c)} \land \text{FrontOf}(a, c) \]

(b) \[ (\text{Cube(b)} \land \text{LeftOf}(a, b)) \lor (\text{Cube(b)} \land \text{RightOf}(a, b)) \land \text{Dodec(c)} \land \text{FrontOf}(a, c) \]

Are sentence (a) and (b) logically equivalent? To determine this, you need to see whether they always have the same truth value in any Tarski’s world. If they always have the same truth value in any Tarski’s world, then they are logically equivalent (Question: How can you do this??). If they do not have the same truth value in one Tarski’s world, then they are not logically equivalent (an counterexample).

1.2 In any formal language

(a) \[ \neg (P(a) \land Q(a)) \]

(b) \[ \neg P(a) \land \neg Q(a) \]

Are the above sentences in FOL logically equivalent? To find out, let us give a possible interpretation to the predicates and names under considerations.

Suppose:

P(x): x is a student.
Q(x): x is a female.
a: Sean

Under this interpretation, sentence (a) means that it is not the case that Sean is a female student or Sean is not a female student (but Sean may be a student). Sentence (b) says that Sean is not a female, and Sean is not a student either. Suppose further that Sean is a male student in a possible world. Then under the above interpretation and in the above circumstance, sentence (a) is true but sentence (b) is false.

In conclusion, sentences (a) and (b) are not logically equivalent (since we have found one possible world in which they do not have the same truth value).

2. How to test logical equivalency?

There are many different formal methods to test for logical equivalency. We will only introduce two of them at this stage.
2.1 Truth-table method
Make a truth table for both sentences to be tested. If they have the same truth values at all rows, then they are logically equivalent. Otherwise they are not.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>¬(P ∧ Q)</th>
<th>¬P ∨ ¬Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Conclusion: “¬(P ∧ Q)” and “¬P ∨ ¬Q” are logically equivalent. That is, 

\[¬(P ∧ Q) \iff ¬P ∨ ¬Q\]

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>¬(P ∧ Q)</th>
<th>¬P ∧ ¬Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Conclusion: “¬(P ∧ Q)” and “¬P ∧ ¬Q” are not logically equivalent.

2.2 Rules of logical equivalency
Rules:
(1) Double negation:  \[¬¬P \iff P\]

(2) De Morgan Rules:  
\[¬(P ∧ Q) \iff ¬P ∨ ¬Q\]  
\[¬(P ∨ Q) \iff ¬P ∧ ¬Q\]

- Negation normal form: “¬” only applies to atomic sentences.

(3) Idempotence:  
\[P ∧ P \iff P\]  
\[P ∨ P \iff P\]

(4) Commutative rules:  
\[P ∧ Q ∧ R \iff R ∧ P ∧ Q\]  
\[P ∨ Q ∨ R \iff Q ∨ R ∨ P\]

(5) Association rules:  
\[(P ∧ Q) ∧ R \iff P ∧ (Q ∧ R)\]  
\[P ∨ (Q ∨ R) \iff (P ∨ Q) ∨ R\]

(6) Distribution rules:  
\[P ∧ (Q ∨ R) \iff (P ∧ Q) ∨ (P ∧ R)\]  
\[P ∨ (Q ∧ R) \iff (P ∨ Q) ∧ (P ∨ R)\]

- Disjunctive normal form: disjunction of conjunction of literals.  
- Conjunctive normal form: conjunction of disjunction of literals.
Examples:

(1) Put $\neg [(A \land B) \land (C \lor \neg D)]$ into negation normal form.

$$\neg [(A \land B) \land (C \lor \neg D)]$$
$$\iff \neg (A \land B) \lor \neg (C \lor \neg D)$$
$$\iff (\neg A \lor \neg B) \lor \neg (C \land D)$$

(2) Put $(A \lor B) \land C \land [\neg (\neg A \land \neg B) \lor B]$ into negation normal form

$$(A \lor B) \land C \land [\neg (\neg A \land \neg B) \lor B]$$
$$\iff (A \lor B) \land C \land [B \lor A \lor B]$$
$$\iff (A \lor B) \land C \land (B \lor A \lor B)$$
$$\iff (A \lor B) \land C \land (A \lor B)$$

(3) Put $(A \lor B) \land (C \lor D)$ into disjunctive normal form.

$$(A \lor B) \land (C \lor D)$$
$$\iff [(A \lor B) \land C] \lor [(A \lor B) \land D]$$
$$\iff [(A \land C) \lor (B \land C)] \lor [(A \land D) \lor (B \land D)]$$
$$\iff (A \land C) \lor (B \land C) \lor (A \land D) \lor (B \land D)$$

(4) Prove that $\neg (A \lor B) \land \neg (B \lor C)$ and $\neg A \land \neg B \land \neg C$ are logically equivalent. Prove:

$$\neg (A \lor B) \land \neg (B \lor C)$$
$$\iff \neg A \land \neg B \land \neg B \land \neg C$$
$$\iff \neg A \land \neg B \land \neg C \land \neg C$$

(5) Prove that $(A \lor B) \land C \land \neg (\neg B \land \neg A)$ and $(A \lor B) \land C$ are logically equivalent. Prove:

$$(A \lor B) \land C \land \neg (\neg B \land \neg A)$$
$$\iff (A \lor B) \land C \land \neg (\neg B \land \neg A)$$
$$\iff (A \lor B) \land C \land (B \land A)$$
$$\iff (A \lor B) \land C \land (B \land A)$$

as desired!
1.3.3. Translation (3.6 of the Text)

1. A standard of a correct translation

A (not the) logical symbolization of an English sentence is correct iff both are logically equivalent.

2. Tips for translations involving conjunctions, disjunctions and negations

(1) Identify the primary connective of the original English sentence to be translated.

Example:

Both d and c are cubes; moreover neither of them is small.

Primary

(Cube(d) ∧ Cube(c)) ∧ (¬ Small(d) ∧ ¬ Small(c))

(2) Paraphrase, if necessary, the sentences to be translated before translation (as long as they are both logically equivalent).

Example:

The original sentence: Neither e nor a is to the right of c and to the left of b.

Rewrite the sentence as:

“Neither e is to the right of c and to the left of b nor a is to the right of c and to the left of b.”

“It is not the case that e is to the right of c and to the left of b AND it is not the case that a is to the right of c and to the left of b.”

After the paraphrasing, translation is easy:

¬ [RightOf(e, c) ∧ LeftOf(e, b)] ∧ ¬ [RightOf(a, c) ∧ LeftOf(a, b)]

(3) A few common patterns of sentences

• Exclusive sense of OR, P or Q (but not both): (P ∨ Q) ∧ ¬ (P ∧ Q)
• Neither P not Q ⇔ It is not that either P or Q: ¬ P ∧ ¬ Q ⇔ ¬ (P ∨ Q)
• P unless Q: P ∨ Q
3. Class exercises

Problem 17 (p. 51)

A translation manual:

<table>
<thead>
<tr>
<th></th>
<th>English</th>
<th>FOL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Names</td>
<td>Abe, AIDS, Al, Bill, Daisy, Dan, Dee, George, Influenza, Monday, Polonius, Stephen, Sunday,</td>
<td>the same</td>
</tr>
<tr>
<td>Predicates</td>
<td>x admires y</td>
<td>Admires(x, y)</td>
</tr>
<tr>
<td></td>
<td>x is a borrower.</td>
<td>Borrower(x)</td>
</tr>
<tr>
<td></td>
<td>x fooled y on z</td>
<td>Fooled(x, y, z)</td>
</tr>
<tr>
<td></td>
<td>x is jolly</td>
<td>Jolly(x)</td>
</tr>
<tr>
<td></td>
<td>x is a lender.</td>
<td>Lender(x)</td>
</tr>
<tr>
<td></td>
<td>x is less contagious than y</td>
<td>LessContagious(x, y)</td>
</tr>
<tr>
<td></td>
<td>x lives near y</td>
<td>Lives(x, y)</td>
</tr>
<tr>
<td></td>
<td>x is a miller.</td>
<td>Miller(x)</td>
</tr>
<tr>
<td></td>
<td>x is more deadly than y</td>
<td>MoreDeadly(x, y)</td>
</tr>
<tr>
<td></td>
<td>x is a river.</td>
<td>River(x)</td>
</tr>
<tr>
<td>Functions</td>
<td>the eldest child of x</td>
<td>eldestChild(x)</td>
</tr>
</tbody>
</table>

Translation:
1. LessContagious(AIDS, Influenza) \( \land \) MoreDeadly(AIDS, Influenza)
2. Fooled (Abe, Stephen, Sunday) \( \land \) \neg Fooled(Abe, Stephen, Monday)
3. (Admire(Dan, Al) \( \land \) Admires(Dan, Bill)) \( \lor \) (Admires(George, Al) \( \land \) Admires(George, Bill))
4. Jolly(Daisy) \( \land \) Miller(Daisy) \( \land \) River(Dee) \( \land \) Lives(Daisy, Dee)
5. \neg (Borrower(eldestChild(Polonius)) \( \lor \) Lender(eldestChild(Polonius)))
1.3.4. Formal Proofs (3.9 of the Text)

1. Derivation rules for conjunctions, disjunctions, and negations

1.1 Simple Rules (without subproofs)

(1) Conjunction Elimination (\(\land\) Elim)

\[
\begin{array}{c}
\frac{\text{m} \quad P \land R \land Q}{\text{\(\land\) Elim: \#m}}
\end{array}
\]

Example (a): \(\{A \land B, C\} \vdash B \land C\)

Example (b): \(\{P \lor Q, R\} \vdash R \land (P \lor Q)\)

(2) Conjunction Introduction (\(\land\) Intro)

\[
\begin{array}{c}
\frac{\text{m} \quad P \quad \text{n} \quad R \quad \text{l} \quad Q}{\text{\(\land\) Intro: \#m. \#n, \#l}}
\end{array}
\]

(3) Disjunction Introduction (\(\lor\) Intro)

\[
\begin{array}{c}
\frac{\text{m} \quad P}{\text{\(\lor\) Intro: \#m}}
\end{array}
\]

Example (c): \(\{P \land Q\} \vdash Q \lor R\)

Example (d): \(\{\neg\neg(Q \land R), P\} \vdash (R \land P) \lor Q\)

(4) Negation Elimination (\(\neg\) Elim)

\[
\begin{array}{c}
\frac{\text{m} \quad \neg P}{\text{\(\neg\) Elim: \#m}}
\end{array}
\]

Example (e): \(\{\neg\neg(Q \land R), P\} \vdash (R \land P) \lor Q\)
1.2 Complex Rules (with subproofs)

(5) **Disjunction Elimination**

\[
\begin{array}{c}
\#l \quad P \lor Q \\
\#n, \quad P \\
\quad \vdots \\
\quad #n, \quad R \\
\quad \vdots \\
\#m, \quad Q \\
\quad \vdots \\
\quad #m, \quad R \\
\quad \vdots \\
R \lor \text{Elim: } #l, #n, \neg #n, \neg #m \\
\end{array}
\]

Two illustrations:
First, let us consider the so-called Disjunctive Dilemma: Protagoras vs. Euathlus
(a) Euathlus will either lose or win.
(b) If he loses the case, then he has to pay back my tuition (by the order of the court).
(c) If he wins the case, then he has to pay back my tuition also (by the terms of the contract).
(d) Either way, Euathlus has to pay back my tuition.

That is,
(a) Lose(Euathlus) \lor Win (Euathlus)
(b) Lose (Euathlus) \rightarrow PayBack(Euathlus, Tuition)
(c) Win (Euathlus) \rightarrow PayBack (Euathlus, Tuition)
(d) PayBack (Euathlus, Tuition)

Secondly, suppose that we want to prove

\{(Cube(c) \land Small (c)) \lor (Tet(c) \land Small (c))\} \models Small (c)

To prove it, let us break it into two cases, corresponding to the two disjuncts as follows:

\[
\begin{array}{c}
1. \quad \text{Cube(c) } \land \text{ Small (c)} \\
2. \quad \text{Small (c)} \\
3. \quad \text{Small (c) } \land \text{ Elim: 1} \\
\end{array}
\]

\[
\begin{array}{c}
1. \quad \text{Tet(c) } \land \text{ Small (c)} \\
2. \quad \text{Small (c)} \\
3. \quad \text{Small (c) } \land \text{ Elim: 1} \\
\end{array}
\]

There are only two alternatives. And in either case, we have Small (c). Then we have proved that Small (c) is a logical consequence of the premises.
Example (e): \[(A \wedge B) \lor (C \wedge D)\] \[\vdash B \lor D\]

```
1. (A \wedge B) \lor (C \wedge D)
   - 2. A \wedge B
   - 3. B \wedge Elim: 2
   - 4. B \lor D \lor Intro: 3
   - 5. C \wedge D
   - 6. D \wedge Elim: 5
   - 7. B \lor D \lor Intro: 6
   - 8. B \lor D \lor Elim: 1, 2-4, 5-7
```

Example (f): \[(A \wedge B) \lor C\] \[\vdash C \lor B\]

```
1. (A \wedge B) \lor C
   - 2. C
   - 3. C \lor B \lor Intro: 2
   - 4. A \wedge B
   - 5. B \wedge Elim: 4
   - 6. C \lor B \lor Intro: 5
   - 7. C \lor B \lor Elim: 1, 2-3, 4-6
```

(6) **Negation Introduction** (\(\neg\) Intro)

```
\[\neg\neg P\]
```

Illustration: Method of proof by contradiction.
Suppose that we want to prove that \(\neg (b = c)\) is a logical consequence of \(\neg\text{Tet}(c), \text{Tet}(b)\). To prove it, let us ASSUME (for the sake of argument) that \(b = c\), see what will follow from the assumption.

```
1. Tet(b)
2. b = c
3. Tet(c) \ Ind. Id: 1, 2
```

But one premise says that \(\neg\text{Tet}(c)\). This contradicts the logical consequence of our assumption that \(b = c\). Therefore, our assumption that \(b = c\) cannot be true since it leads to a contradiction that \(\neg\text{Tet}(c) \land \text{Tet}(c)\). That means that \(b = c\) in a logical consequence of the original premises. That is what we want to prove.
Example (g): \( \{A\} \vdash \neg\neg A \)

Example (h): \( \{P, \neg P\} \vdash Q \)

1. \( A \)
   2. \( \neg A \)
   3. \( A \land \neg A \land \text{Intro: 1, 2} \)
   4. \( \neg \neg A \neg \text{Intro: 2-3} \)

The moral of example (h): You can prove anything from a contradiction. So you can get whatever you want from a contradiction.

2. How to use subproofs correctly?

2.1 The structure of a subproof

- \( P \) the given premises
- \( : \) the assumption of a subproof
- \( \vdash Q \)
- \( \left\{ \begin{align*} R \\ S \end{align*} \right\} \) a closed subproof
- \( T \)
- \( \left\{ \begin{align*} U \\ \vdash A \\ \vdash B \\ W \end{align*} \right\} \) a closed sub-subproof
- \( \vdash C \)

“C” is what you want to prove (the logical consequence of \( P \)) which is always at the bottom of the main proof and outside any subproof.

2.2 Some features of a subproof

- A subproof begins with an unproved assumption, which can only be used inside the subproof itself, and cannot be used outside a closed subproof.
- Once a subproof has been closed off, it can only be cited as \textit{a whole}. Its individual items are not available anymore.
- A subproof can cite items that occur \textit{earlier outside} the subproof, so long as they do not occur in another subproof that have been closed off.
2.3 An example

Example (i): \{(B \land A) \lor (A \land C)\} \models A \land B ???

1. \((B \land A) \lor (A \land C)\)

2. \(B \land A\)

3. \(B\) \land Elim: 2

4. \(A\) \land Elim: 2

5. \(A \land C\)

6. \(A\) \land Elim: 5

7. \(A\) \lor Elim: 1, 2-4, 5-6

8. \(A \land B\) \land Intro: 7, 3 \textbf{WRONG}!!! “B” is inside a closed off subproof which cannot be used outside.

3. Examples of proofs

3.1 Some tips of proofs

- If premises contain a disjunction, try “\lor\text{Elim}”.
- If premises contain no disjunction, try “\neg\text{Intro}” by negating the conclusion.
- From a contradiction, you can get whatever you want by using “\neg\text{Intro}”.

Suppose that during a proof, you disparately need “R”. Fortunately, there is a contradiction occurring earlier. Then you can get “R” by doing the following:

\[ P \]
\[ \neg P \]
\[ \neg R \]
\[ P \land \neg P \]
\[ \neg R \]
\[ R \]

- Making use of Quasi-disjunction to help to get a contradiction.

\[ \neg(P \lor Q) \]
\[ P \]
\[ P \lor Q \]
\[ (P \lor Q) \land \neg(P \lor Q) \]
\[ \neg P \]
\[ Q \]
\[ P \lor Q \]
\[ (P \lor Q) \land \neg(P \lor Q) \]
\[ \neg Q \]
\[ \neg Q \land \neg P \]
3.2. More examples

Example (j): \{ \phi \} \not\models \neg (P \land Q \land \neg P)  

Example (k): \{ \neg P \lor \neg Q \} \not\models \neg (P \land Q)  \text{ DeMorgan Rule}

\begin{align*}
1. & \phi \quad \text{tip: try "¬Intro"} \\
2. & P \land Q \land \neg P \\
3. & P \land \neg P \quad \land\text{Elim: 3} \\
4. & \neg (P \land Q \land \neg P) \quad \neg\text{Intro: 3-4} \\

1. & \neg P \lor \neg Q \quad \text{tip: try "¬Intro" first.} \\
2. & P \land Q \quad \text{tip: introduce more information} \\
3. & \neg P \\
4. & P \quad \land\text{Elim: 2} \\
5. & P \land \neg P \quad \land\text{Intro: 3, 4} \\
6. & \neg Q \\
7. & Q \quad \land\text{Elim: 2} \\
8. & \neg (P \land \neg P) \\
9. & Q \land Q \quad \land\text{Intro: 6, 7} \\
10. & \neg (P \land \neg P) \quad \neg\text{Intro: 8-9} \\
11. & P \land \neg P \quad \land\text{Elim: 10} \\
12. & P \land \neg P \quad \lor\text{Intro: 1, 3-5, 6-11} \\
13. & \neg (P \land Q) \quad \neg\text{Intro: 2-12} \\
\end{align*}

Example (l): \{ \neg (P \land R) \} \not\models \neg P \lor \neg R  \text{ DeMorgan Rule}

\begin{align*}
1. & \neg (P \land R) \\
2. & \neg (\neg P \lor \neg R) \quad \text{tip: try "¬Intro" first (goal: try to get a contradiction)} \\
3. & \neg P \\
4. & \neg P \lor \neg R \quad \lor\text{Intro: 3} \\
5. & (\neg P \lor \neg R) \land (\neg P \lor \neg R) \quad \land\text{Intro: 2, 4} \\
6. & \neg P \quad \neg\text{Intro: 3-5} \\
7. & P \quad \land\text{Elim: 6} \\
8. & \neg R \\
9. & \neg P \lor \neg R \quad \lor\text{Intro: 8} \\
10. & (\neg P \lor \neg R) \land (\neg P \lor \neg R) \quad \land\text{Intro: 2, 9} \\
11. & \neg R \quad \neg\text{Intro: 8-10} \\
12. & R \quad \land\text{Elim: 11} \\
13. & P \land R \quad \land\text{Intro: 7, 12} \\
14. & (P \land R) \land (P \land R) \quad \land\text{Intro: 13, 1} \\
15. & \neg (P \lor \neg R) \quad \neg\text{Intro: 2-14} \\
16. & \neg P \lor \neg R \quad \neg\text{Elim: 15} \\
\end{align*}