GEOMETRY OF THE JULIA SET

FOR SOME MAPS WITH INVARIANT CIRCLES

A Thesis in
Mathematics

by
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Abstract

We work with a subset of the Julia set generated by Herman’s Blaschke product, \( f_\theta(z) = e^{2\pi i \tau(\theta)} \frac{2(z-3)}{1-3z}, \) for \( \theta \) irrational. The number \( \tau(\theta) \) is chosen to give the map \( f_\theta \) the rotation number \( \theta \) when restricted to the unit circle centered at the origin. The subset we work with is \( J_\theta = J_{f_\theta} \setminus \bigcup_{n=0}^{\infty} (f^{-n}(D)), \) the boundary of the immediate basin of infinity, where \( D \) is the open unit disk centered at zero. The set \( J_\theta \) can be related to the Julia set of \( e^{2\pi i \theta} z + z^2 \) for irrationals of bounded type.

In 1996, C. McMullen showed porosity for the Julia set of \( e^{2\pi i \theta} z + z^2 \) for \( \theta \) an irrational of bounded type. Using similar techniques, I have shown:

**Theorem.** [K. R., 2002]

The boundary of the basin of infinity, \( J_\theta \), is non-uniformly porous for all irrational \( \theta \in (0, 1) \).

Non-uniform porosity is a stronger condition than measure zero. For example, the set of rational numbers has measure zero, but it fails to be non-uniformly porous. The fact that \( J_\theta \) has measure zero is not new— it was shown for irrationals of bounded type by C. L. Peterson in 1996 and for all irrational \( \theta \) by M. Yampolsky in 1999, using the complicated techniques of complex bounds.
# Table of Contents

List of Figures ............................................................... vi

Acknowledgments ............................................................. vii

Chapter 1. Introduction ...................................................... 1

Chapter 2. Concepts from Complex Dynamics ............................. 4
   2.1 Properties of the Julia and Fatou sets .......................... 5
   2.2 Closest returns and continued fractions ....................... 6
   2.3 The hyperbolic metric ............................................. 6
   2.4 Bounding distortion .............................................. 7
   2.5 Quasiconformal mappings ....................................... 8
   2.6 Dimension ......................................................... 8

Chapter 3. General History of the Porosity and Dimension of Julia Sets 12
   3.1 Results from McMullen’s "Self-similarity of Siegel disks and Hausdorff dimension of Julia sets” [Mc1] ......................................................... 13
   3.2 Results from Peterson’s “Local Connectivity of some Julia sets containing a circle with an irrational rotation” [Pet] ............................... 15
   3.3 Other results involving porosity and Julia sets ............... 16

Chapter 4. Definitions ....................................................... 17
   4.1 Properties of $J_{f_{\theta}}$ ......................................... 17
   4.2 Metric properties .................................................. 21

Chapter 5. Nearby visits to $C(0,1)$ .................................... 23
List of Figures

1.1 $J_{\theta}$ FOR $\theta = \frac{1}{2}(\sqrt{5} - 1)$ FROM [Pet]. ........................................... 2

4.1 THE PREIMAGE $f_0^{-1}(C(0,1))$. ................................................................. 17
4.2 THE JULIA SET $J_{f_{\theta}}$. ............................................................... 19
4.3 $J_{\theta}$ FOR $\theta = \frac{1}{2}(\sqrt{5} - 1)$ FROM [Pet]. ........................................... 20
4.4 THE ORDERING OF THE LIMBS NEAR 1. ................................................. 20

5.1 THE RECTANGLE $R_n$ AND THE CONE $K$. ................................. 25
5.2 THE TRAPEZOID $T$ AND THE SHAVED LIMB $U_0$. ......................... 27
5.3 THE RECTANGLE $B_{(n,k)}$. ................................................................. 29
5.4 THE REGION $W$ AND THE SHAVED LIMB $U_{q_n}$. ............................. 30
5.5 THE REGION $G_n$ AND THE CURVE $\gamma_n$. ........................................ 35

6.1 $J_{\theta}$ FOR $\theta = \frac{1}{2}(\sqrt{5} - 1)$ FROM [Pet]. ........................................... 40
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This thesis was written using a LaTeX document class written by Stephen Simpson. The Julia set, $J_{f^\theta}$, was generated using software of C. McMullen. The set $J_\theta$ was generated by C. L. Peterson and is from [Pet]. The rest of the images were made using Microsoft Visio.
Chapter 1

Introduction

We consider Herman’s Blaschke product \( f_\theta(z) = e^{2\pi i \tau(\theta)} z^2 \frac{z-3}{1-3z} \) for all irrational numbers \( \theta \), where the number \( \tau(\theta) \) is chosen to give the map \( f_\theta \) the rotation number \( \theta \) when we restrict \( f_\theta \) to \( C(0,1) \), the unit circle centered at the origin. The restriction of \( f_\theta \) to \( C(0,1) \) is an orientation-preserving homeomorphism which is conjugate to an irrational rotation.

We can write any irrational number \( \theta \in (0,1] \) uniquely in the form

\[
\theta = \left[\frac{1}{a_1} + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}\right] = [a_1, a_2, a_3 \ldots]
\]

where each \( a_i \) is chosen so \( a_i \geq 1 \). Let \( \frac{p_n}{q_n} = [a_1, a_2, a_3 \ldots a_n] \) be the \( n \)th rational convergent of \( \theta \). An irrational number \( \theta \) is of bounded type or constant type if \( \sup a_i < \infty \).

We consider \( J_\theta \), the boundary of the immediate basin of infinity. This is a subset of the Julia set of \( f_\theta \). Formally, we can construct \( J_\theta \) by the preimages of the interior of the unit disk. In other words, we define \( J_\theta = f_\theta^{-1}(D) \cup \bigcup_{n=0}^{\infty} f_\theta^{-n}(D) \), where \( D \) is the unit disk centered at the origin. See Figure 1.1.

In order to quantify how large a Julia set is, we wish to find its Lebesgue measure and calculate its Hausdorff dimension. Once a set is shown to have Lebesgue measure zero, it is often easier to bound the Hausdorff dimension rather than calculate it directly. One technique for bounding the Hausdorff dimension of the Julia set is to show that the set is porous. A compact set \( \Lambda \) in the complex plane is called porous (or shallow) if there exists a positive integer \( N \) such that, for any \( z \in \Lambda \) and \( 0 < \epsilon < 1 \), when given a square of side length \( \epsilon \) centered at \( z \) and subdivided into \( N^2 \) squares of side length \( \epsilon/N \), at least one of these squares is disjoint from \( \Lambda \).
Fig. 1.1. $J_\theta$ FOR $\theta = \frac{1}{2}(\sqrt{5} - 1)$ FROM [Pet].

Equivalently, a set $\Lambda$ is *porous* if there exists a $0 < K < 1$ such that for all $z \in \Lambda$ and every $0 < r < 1$, there is a ball $B$ of radius $s$, $Kr < s$, disjoint from $\Lambda$ and contained in the ball of radius $r$ about $z$.

Porosity has been shown for the Julia set of the polynomial $P(z) = e^{2\pi i \theta}z + z^2$. This polynomial $P$ is related to our map $F_\theta$ when $\theta$ is an irrational number of bounded type. We can relate this polynomial to Herman’s Blaschke product by the means of quasiconformal surgery, thus relating $J_\theta$ to the quadratic polynomials $P(z)$ for irrationals of bounded type. Each $P$ is conjugate to an irrational rotation near the origin and has an irrationally neutral fixed point at the origin. Therefore $J_P$ has a Siegel disk. For these particular maps, it can be shown that the critical point $c_0$ lies on the boundary of the Siegel disk. McMullen [Mc1] showed for irrational numbers of bounded type that $J_P$ is porous, and thus has Hausdorff dimension less than two.

We have not yet been able to show porosity for $J_{f\theta}$, but we were able to show a weaker condition called non-uniform porosity. A compact set $\Lambda$ is called *non-uniformly porous* if for any
There is a sequence of radii \( \{r_n\} \) with \( \lim_{t \to n} r_n = 0 \) and a constant \( 0 < K < 1 \) such that there is a ball \( B \) of radius \( s, K r_n < s \), disjoint from \( \Lambda \) and contained in the ball of radius \( r_n \) about \( z \).

Using techniques based on McMullen’s [Mc1] proofs, we will show:

**Theorem 1.1.** The boundary of the immediate basin of infinity, \( J_\theta \), is non-uniformly porous for all irrational numbers \( \theta \in (0, 1] \).

Non-uniformly porosity is a stronger condition than having Lebesgue measure zero. For example, the set of rational numbers has measure zero, but fails to be non-uniformly porous. The fact that non-uniform porosity implies Lebesgue measure zero can be seen by showing that non-uniformly porosity at a point implies the point is not a Lebesgue density point. Thus Theorem 1.1 implies \( J_\theta \) has measure zero. The measure of \( J_\theta \) was previously shown to be zero for irrationals of bounded type by C. L. Peterson [Pet] in 1996 and by Yampolsky [Ya] for all irrationals in 1999, using the technique of complex bounds.

Not much is known about the full Julia set \( J_{f_\theta} \). Peterson [Pet] proved that it was locally connected in 1996. We hope that our theorems about \( J_\theta \) will eventually yield more information about \( J_{f_\theta} \).

Chapter 2 will outline some general concepts used from Complex Dynamics. The history of Julia sets and porosity will be outlined in more detail in Chapter 3. Chapter 4 contains definitions and properties related to \( J_\theta \). The main technical theorem will be shown in Chapter 5. Chapter 6 includes the proof of Theorem 1.1. Conclusions and ideas for future work are in Chapter 7.
Chapter 2

Concepts from Complex Dynamics

The Julia set $J_f$ and its complement the Fatou set $F_f$ on the Riemann Sphere, $\hat{\mathbb{C}}$, are named after two of the pioneers of Complex Dynamics, Pierre Fatou and Gaston Julia, who worked in France in the early nineteen hundreds. More details on the inner workings of Complex Dynamics can be found in Carleson and Gamelin [CG] and in Milnor [Mi]. Results and further definitions from Complex Analysis can be found in Ahlfors [Al1].

The most common way to define these sets is to begin with the Fatou set. The Fatou set $F_f$ is the set of all points $z_0 \in \hat{\mathbb{C}}$ such that $\{f^n\}$ is a normal family in some open neighborhood of $z_0$. A sequence of iterates of $f$ is normal on an open set $U$ if every sequence in $\{f^n\}_{n=1}^{\infty}$ has a convergent subsequence, including convergence to infinity. Having a normal family of iterates of $f$ on some neighborhood $U$, is equivalent to the family being equicontinuous on $U$. Once the Fatou set is defined, the Julia set $J_f$ is the complement of the Fatou set. For a rational function of degree greater than or equal to two, the Julia set is non-empty. Equivalently, we can first define the Julia set $J_f$ of a map $f$ first. The Julia set is the closure of the set of repelling periodic points for $f$. Then, of course, the Fatou set $F_f$ is the compliment of the Julia set.

The map $f$ is commonly a polynomial or a rational function. For a polynomial, $P$, the definition of $J_P$ is simpler, it is the boundary of the basin of infinity.

It is often useful to consider the filled in Julia set, $K_f$, which is the Julia set unioned with the bounded components of the Fatou set. Commonly in computer drawn pictures of the Julia set, $K_f$ is the region colored black.

If infinity is a superattracting fixed point (as it is for $f_\theta$), then the immediate basin of infinity is simply-connected. Thus we can call $\phi$ the Riemann mapping conjugating $J_\theta$ and the
immediate basin of infinity and $\to z^2$ on $\hat{\mathbb{C}} \setminus D(0,1)$. An *external ray* is the image of of a line \( \{re^{i2\pi \eta}\} \) for $\eta \in [0,1]$ under $\phi$. A ray lands on a point of the Julia set if the ray has a continuous extension to the boundary. It is known that there are rays that land on all repelling periodic points.

### 2.1 Properties of the Julia and Fatou sets

The points in the Julia set $J$ have many properties. One of the most useful is that given any point $z \in J$ the preimages of $z$ are dense in $J$.

Sullivan’s theorem showed that components of the Fatou set cannot wander, that is that all components of the Fatou set are eventually periodic. Every periodic component $U$ of the Fatou set can be classified by either $U$ contains an attracting periodic point, is parabolic, is a Siegel disk, or is a Herman ring. Since Siegel disks are present in maps related to the one we will consider, we will define them further.

A *Siegel disk* is the simply connected component in which $f$ is conjugate to an irrational rotation. A Siegel disk occurs when we have a irrationally neutral fixed point $z_0$ with multiplier $|f'(z_0)| = \lambda = e^{2\pi i \theta}$, where $\theta$ is an irrational number satisfying the Brjuno condition below. The condition was established and shown sufficient by Brjuno [Br] and shown necessary by Yoccoz [Yoc]. As mentioned earlier, an irrational number $\theta \in (0,1)$ can be uniquely written in the form

$$\theta = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} = [a_1, a_2, a_3 \ldots]$$

where $a_i$ is chosen so $a_i \geq 1$.

Let $\frac{p_n}{q_n} = [a_1, a_2, a_3 \ldots a_n]$ be the $n$th rational convergent of $\theta$. The Brjuno condition is

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < +\infty.$$
2.2 Closest returns and continued fractions

For a map \( f \) on \( C(0,1) \) with an irrational rotation number \( \theta \), we can relate the continued fraction expansion \( \theta \) to the closest return on the circle of the image of the critical point, which for our \( f_\theta \) is at 1.

In the \( n \)th rational convergent of the continued fraction expansion of \( \theta \), \( q_n \) and \( p_n \) are related to \( a_n \) by \( q_n = a_n q_{n-1} + q_{n-2} \) and \( p_n = a_n p_{n-1} + p_{n-2} \). General information about continued fractions can be found in [HW].

Let the preimages of the critical point be denoted \( x_k = f^{-k}(1) \), and let the images of the critical point, the critical values, be denoted \( c_k = f^k(1) \). It is easiest to consider these on the lifting to the interval \((-\frac{1}{2}, \frac{1}{2}]\) instead of on the circle, on which 0 represents the critical point. We define the closest return to the critical point to be \( f^{-n}(0) \), such that for all \( 0 < k < n \), \(|f^{-k}(0) - 0| > |f^{-n}(0) - 0|\).

We can order the closest returns in the following manner:

\[ x_{q_{n-1}} < 0 < x_{q_n} < c_{q_{n-1}} < x_{(a_n-1)q_{n-1}+q_{n-2}} < \cdots < x_{q_{n-1}+q_{n-2}} < x_{q_{n-2}}. \]

We will see that in our case the \( x_n \) will play the role of the intersection points of \( C(0,1) \) with the other portions of \( J_\theta \).

2.3 The hyperbolic metric

Any region omitting three values from the Riemann sphere is hyperbolic and therefore can be endowed with the hyperbolic metric. Let \( d_{\rho_r} \) be the hyperbolic metric for a hyperbolic region \( R \). The reason for wishing to use this metric is the following theorem, which is from [CG].
Theorem 2.1. Suppose $f$ maps a hyperbolic Riemann surface $R$ holomorphically into a hyperbolic surface $S$. Then

$$f^*(d\rho_S) \leq d\rho_R,$$

$$\rho_S(f(z_1), f(z_2)) \leq \rho_R(z_1, z_2),$$

for $z_1, z_2 \in R$, with strict inequality unless $f$ lifts to a Möbius transformation mapping the unit disk centered at the origin onto itself.

An often used corollary of this Theorem is that if $f$ maps its domain into a subset of itself, $f$ is contracting in the hyperbolic metric. This implies that $f^{-1}$ is expanding. The hyperbolic metric will be comparable to the Euclidean metric in our case. Thus, knowing that the hyperbolic metric is expanding will be useful.

2.4 Bounding distortion

A function $f$ is said to have bounded distortion on some set $X$ if there is a $K > 1$ such that for all $x, y \in X$

$$\frac{1}{K} \leq \frac{|f(x) - f(y)|}{|x - y|} \leq K.$$

The Koebe one-quarter theorem and the Koebe distortion theorem [Mc2] are common tools for bounding distortion. We define $S$ to be the set of univalent functions $f : D(0,1) \to \hat{\mathbb{C}}$ with $f(0) = 0$ and $f'(0) = 1$.

Theorem 2.2 (Koebe one-quarter theorem). Let $f \in S$ be given. Then the image of the unit disk centered at the origin under $f$ contains the open disk of radius $\frac{1}{4}$ centered at the origin.

Theorem 2.3 (Koebe distortion [Mc2]). For $r < 1$, $x$ and $y \in D(0,r)$ and $f \in S$ we have

$$\frac{1}{C(r)} \leq \frac{|f(x) - f(y)|}{|x - y|} \leq C(r).$$
and

\[
\frac{1}{C(r)} \leq |f'(x)| \leq C(r)
\]

where \( C(r) \to 1 \) as \( r \to 0 \).

## 2.5 Quasiconformal mappings

A good resource for information on these functions is Lehto and Virtanen [LV] and Ahlfors [Al2].

A mapping \( f \) is quasiconformal in a region \( D \), if the Beltrami coefficient of \( f \), \( \mu = \frac{f_z}{f_{\bar{z}}} \), satisfies \( |\mu| \leq k < 1 \) for all \( z \in D \). Quasiconformal mappings take ellipses to circles.

A quasicircle is a the image of a circle under a quasiconformal homeomorphism of the plane. A quasidisk is a the image of a disk under a quasiconformal homeomorphism of the plane.

## 2.6 Dimension

Although there are many notions of dimension, we will discuss the Hausdorff dimension, and the upper and lower box dimensions. A good reference for these concepts is Pesin's book, [Pes].

To define Hausdorff dimension, we must begin with the definition of the \( \alpha \)-Hausdorff measure of a set \( Z \) in an Euclidean metric space, \( m_H(Z, \alpha) \). First, define

\[
M(Z, \alpha, \epsilon) = \inf_{\{U_i\}} \left\{ \Sigma_{i=1}^{\infty} (\text{diam}U_i)^\alpha : Z \subset \cup_{i=1}^{\infty} U_i, \text{diam}U_i \leq \epsilon \right\}.
\]

Then

\[
m_H(Z, \alpha) = \lim_{\epsilon \to 0} M(z, \alpha, \epsilon)
\]

is an outer measure on our space and so induces the \( \alpha \)-Hausdorff measure.
Using the $\alpha$-Hausdorff measure, we define the Hausdorff dimension of a set $Z$ as

$$\dim_H(Z) = \inf\{\alpha : m_H(Z, \alpha) = 0\} = \sup\{\alpha : m_H(Z, \alpha) = \infty\}.$$ 

Using similar methods, we can define an upper and lower box measure and hence an upper and lower box dimension. We do this by defining

$$B(Z, \alpha, \epsilon) = \inf\left\{ \sum_{i=1}^{\infty} (diamU_i)^\alpha : Z \subset \bigcup_{i=1}^{\infty} U_i, diamU_i = \epsilon \right\}.$$

Then the upper box measure is generated from the outer measure $r(Z, \alpha) = \lim_{\epsilon \to 0} B(Z, \alpha, \epsilon)$. The lower box measure $\tau(Z, \alpha)$ differs only in that we take the lower limit instead of the upper limit.

Once we have these measures, the upper box dimension is

$$\overline{\dim}_B(Z) = \inf\{\alpha : \tau(Z, \alpha) = 0\} = \sup\{\alpha : \tau(Z, \alpha) = \infty\}.$$ 

The lower box dimension $\underline{\dim}_B$, differs from the upper box dimension in that we take a lower limit.

For upper and lower box dimension there is an equivalent definition more useful for calculations. The **upper box dimension** of a set $\Lambda$ is:

$$\overline{\dim}_B \Lambda = \lim_{r \to 0} \frac{\log N(\Lambda, r)}{\log \left( \frac{1}{r} \right)},$$

where $N(\Lambda, r)$ is the minimum number of boxes of radius $r$ required to cover $\Lambda$, and the radius of a box is its side length. It is equivalent to use balls of radius $r$. We can define the lower box dimension in a similar manner by using a lower limit instead of an upper limit.
As we can see from the original definitions,

\[ \dim_H(Z) \leq \overline{\dim}_B(Z) \leq \underline{\dim}_B(Z). \]

Frequently the Hausdorff dimension, the most difficult to calculate, is estimated by the box dimension.

When calculating or estimating the upper or lower box dimension of a set, we frequently would like to consider a subsequence of box coverings of our set and need conditions by which this gives us the box dimension. From [Pes] we have

**Theorem 2.4 (Box Counting Theorem).** Assume that \( \epsilon_n \) is a monotonically decreasing sequence of numbers, \( \epsilon_n \to 0 \), and \( \epsilon_{n+1} \geq C\epsilon_n \), for some \( C > 0 \) which is independent of \( n \). Assume there also exists the limit

\[ \lim_{n \to \infty} \frac{\log N(\Lambda, \epsilon_n)}{\log \left( \frac{1}{\epsilon_n} \right)} = d. \]

Then \( \overline{\dim}_B(Z) = \underline{\dim}_B(Z) = d. \)

The Box Counting Theorem allows us to consider sequences of boxes rather than having to take the limit over all coverings. This does of course assume that we can find such a sequence.

The main place we use the Box Counting Theorem is in the proof that a porous set on the Riemann sphere has upper box dimension less than 2. This can actually be stated in more generality. A porous set in an \( n \) dimensional Euclidean space has upper box dimension less than \( n \).

**Theorem 2.5.** A porous set \( \Lambda \in \hat{\mathbb{C}} \) has \( \overline{\dim}_B(\Lambda) < 2 \)

**Proof.** Pick any \( r \),

such that \( 0 < r < 1 \). Take any cover of \( \Lambda \) of boxes with side length \( r \). Since \( \Lambda \) is compact, there is a finite subcover of \( m \) boxes.
Divide each box in the cover into $N^2$ boxes of side length $\frac{r}{N}$, where $N$ is the constant of porosity. Since $\Lambda$ is porous, at least one of these $N^2$ boxes is disjoint from $\Lambda$, so we only need at most $N^2 - 1$ of these boxes for each original box in the cover. We can write $N^2 - 1$ as $N^d$ for some number $d < 2$.

So now we have a cover of $mN^d$ boxes of side length $\frac{r}{N}$. Repeat the previous process of subdivision and eliminating unnecessary boxes. Now we have $mN^{2d}$ boxes of side length $\frac{r}{N^2}$. After repeating this process $k$ times, we have $mN^{Kd}$ boxes of side length $\frac{r}{N^k}$. This is a subsequence of covers and radii which satisfy the Box Counting Theorem.

So we have

$$\dim_B(\Lambda) \leq \lim_{k \to \infty} \frac{\log (mN^{kd})}{\log (N^{k}/r)} = d < 2$$
Chapter 3

General History
of the Porosity and Dimension of Julia Sets

The calculation or bounding of Hausdorff dimension is one way to quantify the complexity of a set with Lebesgue measure zero. Given a porous set in an n-dimensional Euclidean space, one can show that the upper-box dimension and hence the Hausdorff dimension are less than n. If the constants of porosity can be calculated exactly, we can extend the bound to $\dim_H(E) \leq n - Ce^n$, where $c$ is the constant that comes from the porosity of $E$ and $C$ only depends on $n$, as shown in [MV].

The concept of porosity has been extended by Koskela and Rohde in [KR] to a notion of mean porosity. To do this for a bounded set $E$ in $\mathbb{R}^d$, $\epsilon > 0$; and $c \leq q$, they defined

$$A_n(x) = \{y \in \mathbb{R}^d : (1 + \epsilon)^{-n} < |x - y| < (1 + \epsilon)^{-n+1}\}$$

and

$$\chi_n(x) = \begin{cases} 1 & \text{there exists } y \in A_n(x) \text{ such that } D(y, E) > ce|x - y|, \\ 0 & \text{otherwise}. \end{cases}$$

They also defined

$$s_n(x) = \sum_{k=1}^{\infty} \chi_k(x).$$
The set $E$ is \textit{mean $\epsilon$-porous} if there is an integer $n_0$ such that \( \frac{s_n(x)}{n} > \frac{1}{2} \) for all $x \in E$ and all $n \geq n_0$. $E$ is \textit{weak mean $\epsilon$-porous} if
\[
\liminf_{n \to \infty} \frac{s_n(x)}{n} > \frac{1}{2}
\]
for each $x \in E$. Koskela and Rohde showed
\[
dim_H(E) \leq d - C\epsilon^{d-1}
\]
if $E$ is a weakly mean $\epsilon$-porous set, where $C$ is a constant only dependent on the dimension $d$ and $c$ and $C$ satisfies $C \geq C(d)c^d$. If $E$ is a mean $\epsilon$-porous set the same statement can be made about the Minkowski dimension. One of our future goals is to see if mean porosity is implied by non-uniform porosity. We will discuss this at greater length later.

Using porosity to bound dimension is not a new. As Przytycki and Urbański say in [PU], “For rational functions expanding on a Julia set the proof of porosity is easy (it was folklore since a long time). Just pull-back large scale holes to all small scales by iteration of inverse branches of $f$.” There have been quite a few successful attempts to show certain classes of Julia sets are porous. One was by McMullen [Mc1] in 1996. Since most of the results in our future chapters follow from his techniques, we will discuss his results further.

\subsection{3.1 Results from McMullen’s ”Self-similarity of Siegel disks and Hausdorff dimension of Julia sets” [Mc1].}

In 1996, C. McMullen showed porosity for the Julia set of $e^{2\pi i \theta}z + z^2$ for $\theta$ an irrational of bounded type. In order to show porosity and hence bound the Hausdorff dimension, we would like a way to replicate behavior near a point in $J_P$ at any other point. In this case we wish to replicate behavior near the critical point $c_0$. This will allow us to propagate behavior
near the critical point, with bounded distortion, near any point in $J_P$. We define the inner-radius $(U, y) = \sup \{ r : B(y, r) \subset U \}$ and state the following result of McMullen:

**Theorem 3.1 (Nearby Critical Visits).** For every $z \in J_P$ and $r > 0$, there is a univalent map between pointed disks of the form $P_i : (U, y) \to (V, c_0)$, $i \geq 0$ such that $\frac{r}{K} \leq \text{inner-radius}(U, y) \leq Kr$ and $|y - z| \leq Cr$, where $C$ and $K$ are constants that depend only on $\theta$.

Thus the Nearby Critical Visits Theorem gives a univalent mapping from a ball near a given point in $J_P$ to a neighborhood of the critical point $c_0$. We can move balls near the critical point to balls near any other point using the Koebe distortion theorem. Hence, behavior near the critical point can be replicated everywhere.

The proof of the Nearby Critical Visits Theorem consists of three cases. The first case is when a point $z \in J_P$ is close to the Siegel disk. The properties of the conjugacy with $f_\theta$ and the fact that $\theta$ is of bounded type are used to establish the map which McMullen calls an approximate rotation. The second case is when a neighborhood of $z$ can be mapped near the Siegel disk with bounded distortion. We establish the map by moving near the Siegel disk and applying the first case. The third case is for all other points in $J_P$. We use the fact that $J_P$ is contained in the region where the hyperbolic injectivity radius on $\Omega$ is bigger than one to establish a univalent map of hyperbolic balls. After this, the comparison between the Euclidean metric and the hyperbolic metric is used to get a univalent mapping on Euclidean balls.

Once we have this replication, the proof of porosity follows quickly. By the Herman-Świątek theorem, the Siegel disk is a quasicircle. Thus we can find a ball centered at $C_0$ which is disjoint from $J_f$ inside $D$ at any scale. We take these holes, use the replication result, and the Koebe Distortion Theorem to move the holes with bounded distortion near any point in $J_f$.

We will use technique similar to that used in McMullen’s third case to get mappings of hyperbolic balls near all but a countable number of the limit points. We have the benefit of being
able to map to any point on the circle, not just the critical point and the detriment of not having the uniformity given by considering only irrational numbers of bounded type.

### 3.2 Results from Peterson’s “Local Connectivity of some Julia sets containing a circle with an irrational rotation” [Pet]

This paper contained results about the sets $J_\theta$, $J_{f\theta}$, and $J_P$, most of which were about local connectivity. A compact set $\Lambda$ is \textit{locally connected} at a point $z$ if for every sequence $\{z_n\}$ converging to $z$, there is sequence of connected sets $L_n \subset \Lambda$ containing both $z$ and $z_n$, such that the diameter of $L_n$ goes to zero as $n \to \infty$. A compact set $\Lambda$ is locally connected if all of its points are locally connected. We study local connectivity because it allows us to extend the Riemann mapping. The Carathéodory Theorem states that if a simply connected set $\Lambda$ is locally connected and its boundary contains at least two points, then the Riemann mapping to $\Lambda$ extends to the boundary.

The main results from Peterson’s paper are:

**Theorem 3.2.** For every $\theta$ of bounded type, the Julia set $J_P$ is locally connected and has zero Lebesgue measure.

**Theorem 3.3.** For any irrational $\theta \in (0, 1) - \mathbb{Q}$, $J_{f\theta}$ and $J_\theta$ are locally connected.

**Theorem 3.4.** For every $\theta$ of bounded type $J_\theta$ has Lebesgue measure zero.

Note that the first theorem follows from the second and third theorems and the Ghys construction.

The main tool of Peterson’s paper that we use is the “suspension bridge” in section 3. Peterson uses the “suspension bridge” to separate the primary limb, $X_0$, from the circle. We will construct a simpler version of his tool in order to separate the primary limb, $X_0$, from the circle, and also in order to pick a cone so that most of $X_0$ eventually is mapped into the cone.
3.3 Other results involving porosity and Julia sets

Other results for porosity of Julia sets are for rational maps of the Riemann sphere satisfying a certain Collet-Eckmann condition on their critical points. A map \( f \) satisfies the Collet-Eckmann condition if every critical point \( c \) of \( f \) such that \( c \in J_f \) and that the orbit of \( c \) does not contain any other critical points, satisfies \( |(f^n)'(f(c))| \geq C\lambda^n \) for all \( n \) and some \( C > 0 \) and \( \lambda > 1 \).


In 1997, Järvi [Jä] proved that quasi-self-similar Julia sets are porous. In 1999, Geyer [Ge] showed that the Julia set of all parabolic rational maps are porous. Yongcheng [Yon] showed in 2000 that the Julia set of a critically recurrent rational map is porous. The Hausdorff dimension in the cases considered by Geyer and Yongcheng was already known to be less than two, but from different methods.
Chapter 4

Definitions

To accurately describe portions of the Julia set, we must establish a few definitions. Again, the orbit of the critical point is of importance to us. As before we let the preimages of the critical point 1 be called $x_n = f^{-n}(1)$. In order to describe some other portions related to the $x_n$ on $J_{f_\theta}$ and $J_{\theta}$, we will consider first the properties of $J_{f_{\theta}}$.

4.1 Properties of $J_{f_{\theta}}$.

We begin with the degree three Blaschke product, $f_0(z) = z^2(z-3)$. The map $f_0(z)$ leaves $C(0,1)$ invariant, and is also symmetric in this circle. It has three critical points: 0, 1, and $\infty$. If we examine $(f_0)^{-1}(C(0,1))$, we note there are two preimages of the circle. The first is the circle itself and the second is a curve passing through the critical point 1. See Figure 4.1.

![Figure 4.1](image-url)
Similarly, we can consider \( f_\theta(z) = e^{2\pi i \tau(\theta)}f_0(z) \), which has the above properties as well. Also, for \( \theta \) irrational, \( C(0, 1) \) belongs to the Julia set of \( f_\theta \). This follows from the fact that the point 1 is in the Julia set, the forward orbit of 1 is dense in \( C(0, 1) \), and the Julia set is forward invariant. Since \( C(0, 1) \) is part of the Julia set, the closure of the preimages of \( C(0, 1) \) must be the whole Julia set.

The boundary of the immediate basin of infinity, \( J_\theta = \bigcup_{n=0}^{\infty} f_\theta^{-n}(D) \), has the property that its points fall in three classes: points on \( C(0, 1) \), points on finite preimages of \( C(0, 1) \), and the points which are limits of preimages of \( C(0, 1) \), which we will refer to as limit points. This property follows from the properties of \( J_\theta \).

Now let’s examine \( f_\theta^{-1}(C(0, 1)) \cap J_\theta \). It now consists of two parts, the first being \( C(0, 1) \) and the second being a teardrop shaped curve, a bulb, intersecting \( C(0, 1) \) at the critical point 1, which we will call the primary preimage of \( C(0, 1) \), \( C' \). In fact all preimages of \( C(0, 1) \) contained in \( J_\theta \), will be made up of such bulbs. The bulbs which intersect \( C(0, 1) \) will do so at the preimage of the critical point \( x_n \) for some \( n \). These bulbs which intersect the circle will be called main bulbs with basepoint \( x_n \).

We consider the point \( x_n \). If we remove the point \( x_n \) from \( J_\theta \), we get two connected components, one of which contains points in \( C(0, 1) \) and the other. We will call the other component unioned with \( x_n \) the limb \( X_n \), with basepoint \( n_n \). We call the limb with the basepoint \( x_0 = 1 \), the primary limb, \( X_0 \). The primary preimage of \( C(0, 1) \) is called \( C' \) or the main bulb of \( X_0 \). The main bulb of a limb \( X_n \) will be the primary \( n \)th preimage of the circle, namely the preimage in \( X_n \) which contains the point \( x_n \).

Note that \( f^n(X_n) = X_0 \). This property allows us to examine behavior on all limbs by examining behavior on the primary limb, \( X_0 \). The basepoints \( x_{q_n} \) are the closest returns to the critical point at 1 and by the properties discussed in 2.2, we have the following ordering seen in Figure 4.4.
Fig. 4.2. THE JULIA SET $J_{f_{\theta}}$. 
Fig. 4.3. $J_\theta$ FOR $\theta = \frac{1}{2} (\sqrt{5} - 1)$ FROM [Pet].

Fig. 4.4. THE ORDERING OF THE LIMBS NEAR 1.
One of the other properties of $J_\theta$ is that it can be related to $J_P$, for $P(z) = e^{2\pi i \theta} z + z^2$.

First let’s examine the properties of $J_P$, which has a Siegel disk $D$.

**Theorem 4.1 (Herman-Świątek, 1987).** The critical point $c_0$ is contained in $\partial D$. The boundary of $D$ is a quasicircle. Also the postcritical set $\{c_0, f(c_0), f^2(c_0), \ldots\} = \partial D$.

One of the most amiable properties of this particular family of Julia sets for $P$ is that every other component of each filled in Julia set is eventually mapped into the Siegel disk. This property is shown by looking at a construction of Ghys [Gh], which relates $P$ to $f_\theta$.

By the methods of quasiconformal surgery, we can relate $J_{f_\theta}$ to the previously defined set, $J_\theta$. This “Julia” set $J_\theta$ is quasiformally equivalent to $J_P$. Thus we define:

\[
F_{\theta} = \begin{cases} 
  f_\theta(z) & |z| \geq 1 \\
  H_{\theta}^{-1}R_{\theta}H_\theta(z) & |z| < 1 
\end{cases}
\]

The conjugacy mapping $H_\theta$ is a quasiconformal extension of the conjugacy to the rotation in $D$. Therefore, we get the ”Julia” set $J_\theta = J_{f_\theta} \setminus \bigcup_{n=0}^{\infty} f_\theta^{-n}(D)$, which is left invariant under $F_\theta$.

Using the Herman-Świątek theorem one can show that $J_\theta$ is quasiconformally equivalent to $J_P$ for all $\theta$ of bounded type. In the identification, the unit disk is identified with the Siegel disk $D$, hence $\partial D$ is a quasicircle.

### 4.2 Metric properties

Frequently, it will be useful to consider the hyperbolic metric on the complex plane minus the unit disk centered at the origin. Let $\Omega$ be the space $\mathbb{C} \setminus D(0,1)$ endowed with the hyperbolic metric $d_\Omega$. If we consider any portion of $\Omega$ minus some disk centered at infinity, $\Omega'$, we can estimate the element of the hyperbolic metric $\rho_\Omega(z) |dz|$ as follows:

**Theorem 4.2 (McMullen [Mc2] 2.3).** For some constant $K > 1$,
We will use the notation $\rho_\Omega$ for the element of the hyperbolic metric on $\Omega$. We will use $d_\Omega(\ldots)$ for the hyperbolic distance on $\Omega$ and $D_z$ for the derivative with respect to the hyperbolic metric and $\|\|_\Omega$ when we require the norm with respect to the hyperbolic metric on $\Omega$. 

\[
\frac{1}{Kd(z, \partial \Omega)} \leq \rho_\Omega(z) \leq K \frac{1}{d(z, \partial \Omega)}.
\]
Chapter 5

Nearby visits to $C(0, 1)$

In order to show $J_\theta$ is non-uniformly porous, for $\theta$ irrational, we show that all but a countable number of points map near to the circle $C(0, 1)$. Since points in $J_\theta$ are either on $C(0, 1)$, are on a finite preimage of $C(0, 1)$, or are limit points, we can deal with each in turn. Points on $C(0, 1)$ and finite preimages if $C(0, 1)$ are certainly not Lebesgue density points, so the only class of points that must be considered are the limit points. We will deal with the limit points by finding univalent mappings with bounded distortion which map a ball near each limit point to a region that contains a point on $C(0, 1)$. We will be able to find these mappings for all but a countable number of limit points, which is enough.

So the main result of this chapter will be to show the following:

**Theorem.** For all but a countable number of limit points $x \in J_\theta$, there is a sequence of positive integers $n_k$, $n_k \to \infty$ such that, for each $n_k$ there is an embedded hyperbolic ball $B_{n_k}$ centered at a point $y_{n_k}$ such that for some ball $B'$ and $p \in C(0, 1)$

$$f^{n_{k+1}} : (B_{n_k}, y_{n_k}) \to (B', p).$$

For the constant $\rho$ from the cone $K$ and a constant $\kappa$ depending on $\rho$, \(1/K \leq ||D_z(f^{-n_k})(x)||_\Omega \leq \frac{\text{radius}_\Omega(B_{n_k})}{\rho} \leq \kappa ||D_z(f^{-n_k})(x)||_\Omega, \text{ and } d_\Omega(x, y_{n_k}) < 2\rho \kappa ||D_z(f^{-n_k})(x)||_\Omega.\) The mapping $f^{n_k}$ is univalent and has bounded distortion on $B_{n_k}$.

The fact that $J_\theta$ has measure zero for all irrational $\theta$ will follow shortly from Theorem 5.7, as will the non-uniform porosity proved in the next chapter.
We will define a cone \( K = \{ z \in \Omega \mid d_\Omega (z, C') \leq \rho \} \), where \( C' \) is the primary preimage of \( C(0, 1) \) and \( \rho \) is a constant that will be chosen later. Once \( K \) is defined, we will define some related regions, \( R_n \). This cone will be the place where we are guaranteed to have the mappings for the above theorem. Since each point in \( C' \) maps to \( C(0, 1) \) and a ball of finite hyperbolic radius centered at a point in \( C' \) maps conformally onto \( C(0, 1) \), if we can map all other points in \( J_\theta \) into the cone, we will have similar mappings.

Since inside the cone \( K \) we have our mappings, we want all points to eventually enter the cone. What we show is that all but a countable number of limit points enter the cone infinitely many times on their orbits.

Let \( R_n \) be the open quadrilateral between \( X_{qn} \) and \( X_{qn-2} \). The four sides of the quadrilateral will be defined as follows. The first side is the segment on \( C(0, 1) \) between \( x_{qn} \) and \( x_{qn-2} \), which does not contain 1. Let \( l_n \) be the line segment between \( x_{qn} \) and \( \infty \). The second side of \( R_n \) is the piece of \( l_n \) between \( x_{qn} \) and the point \( y_{qn} \) which is the first intersection of \( l_n \) and the boundary of cone \( K \). Define the third side in the same manner as the second, except using \( x_{qn-2} \). The last side is the segment of the boundary of cone between \( y_{qn} \) and \( y_{qn-2} \). See figure 5.1.

**Proposition 5.1.** For some constant \( \rho \), the cone \( K = \{ z \in \Omega \mid d_\Omega (z, C') \leq \rho \} \) has the following properties: for each \( n \), \( R_n \) contains an open region \( G_n \), such that \( G_n \) contains all points of \( X_0 \) in \( R_n \) and \( f^{qn-1} : G_n \to (G_n \cup K) \).

**Proof.** Let \( n \) be given.

In order to increase clarity we will break the proof into several lemmas. We first show we can pick a region where the height is always a fixed proportion of the length of the dynamical interval. Second, we show this region contains no points of \( X_0 \) for all but a finite number of \( n \). Third, we construct the subregion with the invariant boundary. Once we have these three lemmas we will be able to choose the cone and the region \( G_n \) and verify its properties.
Define a closed box $B_{(n,k)}$, $0 \leq k \leq a_n - 1$; its boundary consists of four sides. The first side is the segment on $C(0,1)$ between $x_{kq_n-1+q_n-2}$ and $x_{(k+1)q_n-1+q_n-2}$, which does not contain 1. Let $l_n$ be the line segment between $x_{kq_n-1+q_n-2}$ and $\infty$. The second side of $B_{(n,k)}$ is the piece of $l_n$ between $x_{kq_n-1+q_n-2}$ and the intersection of $l_n$ with circle of radius $1 + \epsilon_{(n,k)}$ centered at the origin, where $\epsilon_{(n,k)} > 0$ will be chosen later. Define the third side in the same manner as the second, except using $x_{(k+1)q_n-1+q_n-2}$. The last side is the segment of the circle of radius $1 + \epsilon_{(n,k)}$ centered at the origin between the points of intersection of the second and third segments with this circle.

We also define a subregion of $X_{q_n}$ for any $n$. We will do this by choosing a region of the primary limb $X_0$ and taking its preimages.

Let the shaved limb $U_0$ be a curve along $X_0$ which connects 1 to the fixed point on $X_0$. We may choose $U_0$ so that a trapezoid $T$ with an endpoint at 1, an angle of 100 degrees at 1, and a small height $h > 0$ intersects $U_0$ only at 1. Call $U_n = f^{-n}(U)$. 

---

**Fig. 5.1. THE RECTANGLE $R_n$ AND THE CONE $K$.**
Lemma 5.2. We can choose the $\epsilon_{(n,k)}$ so that there exists an $A > 0$ independent of $n$ and $k$ such that $\epsilon_{(n,k)} = A|x(k+1)q_{n-1}+q_{n-2} - xkq_{n-1}+q_{n-2}|$, such that $B_{n,k} \cap U(k+1)q_{n-1}+q_{n-2} = x(k+1)q_{n-1}+q_{n-2}$.

Proof. Consider the regions $B_{(n,k)}$, $0 \leq k \leq a_n - 1$, as above.

We wish to choose $\epsilon_{(n,a_n-1)}$ so that $B_{(n,a_n-1)} \cap U_q = x_q$. Under $f^{qn} B_{(n,a_n-1)}$ maps to a region with one end point at 1. Now we must choose $\epsilon_{(n,a_n-1)}$ to be small enough, so that the image $f^{qn}(B_{(n,a_n-1)})$ is contained in the trapezoid, $T$. See Figure 5.2

We can guarantee this because of the bounded geometry. This comes from the Lemma 1.2 of Świątek from [S], which can be stated in the following manner.
Fig. 5.2. THE TRAPEZOID $T$ AND THE SHAVED LIMB $U_0$. 
Lemma 5.3 (Świątek, [S]). Suppose that $f$ is a real-analytic degree 1 circle homeomorphism with an irrational rotation number. Let $\frac{p_n}{q_n}$ be a convergent of the rotation number. Then there is a $K \geq 1$ so that for all $z \in C(0,1)$,

$$K^{-1}d(f^{q_n}(z), z) \leq d(f^{-q_n}(z), z) \leq Kd(f^{q_n}(z), z).$$

Since our $f$ is a real-analytic circle map with an irrational rotation, we can apply the lemma.

So by Lemma 5.3, if $\epsilon_{(n,a_n-1)}$ is small enough we can map $B_{(n,a_n-1)}$ so that it is contained in the trapezoid $T$. This implies that $f^{q_n}(B_{(n,a_n-1)}) \cap U_0 = 1$, which implies $B_{(n,a_n-1)} \cap U_{q_n} = x_{q_n}$. Since the distortion under $f^{q_n}$ of the box $B_{(n,a_n-1)}$ depends on the length $|x_{q_n} - x_{q_n-q_n-1}|$ and the height of the trapezoid $T$ is fixed, then we can define $\epsilon_{(n,a_n-1)} = A|x_{q_n} - x_{q_n-q_n-1}|$, where $A$ is a constant depending on the height of the trapezoid $T$.

For all other $0 \leq k < a_n-1$, we proceed in the same manner. We choose the height of $B_{(n,k)}$ to be $\epsilon_{(n,k)}$ so that $f^{(a_n-(k+1))q_n-1}(B_{(n,k)})$ is contained in $T$, yielding $B_{n,k} \cap U_{(k+1)q_n-1+q_n-2} = x_{(k+1)q_n-1+q_n-2}$. Then $\epsilon_{(n,k)}$ has the property that

$$\epsilon_{(n,k)} = A|x_{(k+1)q_n-1+q_n-2} - x_{kq_n-1+q_n-2}|$$

for the same reasons that $\epsilon_{(n,a_n-1)} = A|x_{q_n} - x_{q_n-q_n-1}|$.

So in general $B_{(n,k)}$ is a short quadrilateral which is on one side of the shaved limb at $X_{(k+1)q_n-1+q_n-2}$. See Figure 5.3.

Now we examine a $B_{(n,k)}$ for some $k$, $0 \leq k \leq a_n-1$. We want to show that no points of $X_0$ lie in the boxes.
Lemma 5.4. The boxes \( B_{(n,k)} \) contain no points of \( X_0 \) for all \( 0 \leq k \leq a_n - 1 \) and for all but a finite number of \( n \).

Proof. In order to get a contradiction, we will assume there is a point \( z \in (X_0 \cap B_{(n,k)}) \). We will consider the case \( k = a_n - 1 \) first.

We will begin by constructing a region around some part of \( X_0 \) for a \( z \in X_0 \cap B_{(n,a_n-1)} \) and then will repeat a similar construction for \( B_{(n,k)} \). Then we will show these lead to a contradiction.

Given \( z \in X_0 \cap B_{(n,a_n-1)} \), we will define a region relating to \( z \). Take a path \( P \) that connects \( z \) to some point \( z' \) on \( C(0,1) \). Choose the path \( P \), so that is contained in \( B_{(n,a_n-1)} \setminus U_{q_n} \).

We can do this, since \( B_{(n,a_n-1)} \setminus U_{q_n} \) is connected and open and so is path connected. Remember that \( \Omega = \mathbb{C} \setminus D(0,1) \) and \( \Omega \) contains \( U_{q_n} \). Consider the set \( R = \Omega \setminus (P \cup X_0) \), which has several components. Let \( W \) be the closure of the connected component of \( R \) which contains \( U \). Call \( \partial W \) the loop \( L \). See figure 5.4.
Fig. 5.4. THE REGION $W$ AND THE SHAVED LIMB $U_{q_n}$. 
The loop $L$ consists of three pieces. The first is a segment on the circle which contains $x_{qn}$ in the interior. The second is the path $P$. The third is a path on $X_0$ leading to $z$.

Apply $f$ to $W$, and examine what happens to the pieces of the loop. The image of the segment on the circle remains a segment on the circle which contains $x_{qn-1}$ in its interior. The image of the path $P$ remains a short segment connecting with $C(0,1)$. The image of the path on $X_0$ leading to $z$ is a path on some other limb $X_{m_1}$ and some segment of the circle which is connected to the previous segment on the circle. So overall, $f(L)$ consists still of three pieces; a segment of the circle containing $x_{qn-1}$ in its interior which begins at $x_{m_1}$ and ends at $f(z')$, the image of the path $P$, and a path on some limb $X_m$.

By induction, if we apply $f^k$ to $W$ and examine the boundary $f^k(L)$, we have the following three pieces. The first is a segment of $C(0,1)$ containing $x_{qn-k}$ in the interior and the endpoint of the segment are $f^k(z')$ and some basepoint $x_{m_k}$ for some other limb $X_{m_k}$. The second is the image of the path $f^k(P)$. The last is a path along some limb $X_{m_k}$.

Apply $f^{qn}$ to $W$. The image under $f^{qn}$ of the shaved limb $U_{qn}$, which is contained in $W$, is mapped into $U_0$ and it and $f^{qn}(z)$ are contained in $f^{qn}(W)$. The only point of $f^{qn}(U)$ not contained in the interior of $f^{qn}(W)$ is the basepoint 1.

Now examine $f^{qn}(L)$. By the previous argument, $f^{qn}(L)$ consists of a segment of the circle containing 1 in its interior, the image of the path $f^{qn}(P)$, and a path along a limb $X_m$ containing $f^{qn}(z)$. The segment of the circle has endpoints $x_{mqn}$ and $f^{qn}(z')$.

Now we can construct similar regions for any point $z \in B(n,k)$. Using $U_{qn-1+(k+1)q_n-2}$ and using $f^{qn-1+(k+1)q_n-2}$ to map the region $W$ to the primary limb we construct the regions. The property that the image of $z'$ and the basepoint of the image of $z$ are on opposite sides of 1 hold, and the image of the loop still has the three types of pieces. We will use this to achieve a contradiction.

We know that there is an external ray that lands on the point fixed under iteration of $f$ on $X_0$. The preimage of the external ray under $f^{-qn}$ must pass through the loop $L$ and so the
external ray itself must pass through $f^{qn}(L)$. The external ray cannot pass through the circle or any other limb, so the only possible point of intersection is on $P$.

This is only possible for a finite number of $n$ since the length of the image of $P$ must be less than a constant times $\epsilon(n,k)$ for all $k$ and $n$ and $\epsilon(n,k)$ goes to zero as $n$ goes to infinity. If for all $n$ the external ray passes through the image of $P$, then the ray cannot land. This is a contradiction, therefore $B_{(n,k)}$ cannot contain any points of $X_0$ for all but a finite number of $n$ and $k$.

Now that we have a region not containing any points of $X_0$, we wish to obtain an invariant region contained in $\bigcup_{k=1}^{\alpha_n-1} B_{(n,k)}$, so that we have an invariant region containing no points of $X_0$.

We will now define a region contained in $\bigcup_{k=1}^{\alpha_n-1} B_{(n,k)}$. Remember, the second side of $B_{(n,k)}$ is the piece of $l_n$ between $x_{qn-1}+kq_n-2$ and the intersection of $l_n$ and the circle of radius $1+\epsilon(n,k)$ centered at the origin, where $\epsilon(n,k) > 0$ and the third side was determined in the same manner as the second, except using $x_{q_n-1+(k+1)q_n-2}$. Thus the region $\bigcup_{k=1}^{\alpha_n-1} B_{(n,k)}$ has one side with a segment of the ray between $x_{qn}$ and $\infty$ and the other is a segment of the ray between $x_{qn-2}$ and $\infty$.

Pick a point $z$ belonging to the first side of $B_{(n,\alpha_n-1)}$ and a height $\delta < \epsilon(n,\alpha_n-1)$, which will be chosen later. Construct a curve $\gamma_n$ as follows. Its first piece will be the segment between $z$ and $f^{\alpha_n-1}(z)$. Subsequent pieces will be images of the first piece under $f^{\alpha_n-1}$. End the last piece at the point of intersection with the second side of $B_{(n,0)}$.

Let the region $B_n$ be defined by four pieces as follows. The first side will be the segment of $C(0,1)$ between $x_{qn}$ and $x_{qn-2}$ which does not contain 1. The second side will be $\gamma_n$. The third side will be a segment of the ray between $x_{qn}$ and $\infty$, going from $x_{qn}$ to $z$. The last side is
the segment of the ray between \( x_{q_n-2} \) and \( \infty \), going from \( x_{q_n-2} \) to the intersection of \( \gamma_n \) with the ray.

**Lemma 5.5.** The region \( B_n \) contains no points of \( X_0 \) and the top of \( B_n, \gamma_n \), has the property

\[
(f^{q_{n-1}}(\gamma_n) \cap (\bigcup_{k=1}^{q_{n-1}} B_{(n,k)})) \subset \gamma_n.
\]

**Proof.** Notice that \( B_n \) does not contain any points of \( X_0 \), as long as the \( B_{(n,k)} \) do not. So we must choose \( \delta \) so that \( B_n \) is contained in \( \bigcup_{k=1}^{q_{n-1}} B_{(n,k)} \).

The Lemma 5.3 gives us that the distortion from mapping forward, as we did to get the pieces of \( \gamma_n \), is bounded. The bounded distortion implies that \( \delta \)’s value depends only on the heights, \( \epsilon_{(n,k)} \), of the boxes, \( B_{(n,k)} \). Since \( \epsilon_{(n,k)} \) is a fixed proportion of the length of the dynamical interval \([x_{(k+1)q_n-1+q_n-2}, x_{kq_n-1+q_n-2}]\), we can choose \( \delta > 0 \) dependent only on this proportion, so that \( B_n \) is contained in \( \bigcup_{k=1}^{q_{n-1}} B_{(n,k)} \).

Now we have all the tools needed in order to prove the Proposition.

We will choose our cone \( K = \{ z \in \Omega : |d_\Omega(z, C') \leq \rho \} \). Choose \( \rho \) so that two properties are satisfied. First, three pieces of \( \gamma_n \) must be completely inside the cone. These pieces consist of the piece between the starting point \( z \) and \( f^{q_{n-1}}(z) \) and the last two pieces of \( \gamma_n \). If \( \gamma_n \) has three or less pieces, all of \( \gamma_n \) should be inside the cone. Second, the cone \( K \) must contain all of the points of \( X_0 \) in the boxes which do not cause a contradiction in the Loop Lemma 5.4.

The first property can be assured from the way the height \( \delta \) was chosen in Lemma 5.5. The second is feasible since we can lower the cone any finite length needed. Since we can achieve these two properties, we choose \( \rho \) so that properties are satisfied and hence we have defined the cone \( K \).
Note that $f^k(q_n-1)(\gamma_n \cap R_n) \subset \gamma_n \cup K$ for all $k$. Since the only regions of $\gamma_n$ that did not map into $\gamma_n$ are in the cone $K$ and the portion of $\gamma_n$ not in the cone, $\gamma_n \cap R$, maps into itself and the cone by definition. Define $G_n = R_n \setminus B_n$. Now we must verify that $G_n$ maps into $G_n \cup K$ under $f^{q_n-1}$.

Take a connected component of $G_n$. The boundary $\partial G_n$ consists of some portion of the boundary of $K$ and some portion of $\gamma_n$. Since $f^{q_n-1}$ is univalent on $\overline{G_n}$, the boundary maps to the boundary. In fact, from the definition of $\gamma_n$ and the choice of the cone $K$, $f^{q_n-1}(\gamma_n \cap R_n) \subset (\gamma_n \cup K)$. Also since $f^{q_n-1}$ is orientation preserving, the points of $G_n$ near the curve $\gamma_n$ remain in $G_n$ under $f^{q_n-1}$. Thus $G_n$ maps into $G_n \cup K$ under $f^{q_n-1}$.

Now we have the cone $K$ and the region $G_n$ with the appropriate properties, so we are done. \qed

Note that the Proposition 5.1 guarantees that on $G_n$ we have a well defined inverse function $f^{-(q_n-1)}$ from $G_n$ into itself.
Fig. 5.5. THE REGION $G_n$ AND THE CURVE $\gamma_n$. 
**Proposition 5.6.** For all but a countable number of limit points \( z \in J_\theta \). The iterates \( f^n(z) \) are contained in \( K \) for infinitely many \( n \).

**Proof.** Let \( x \in J \) be given. Let \( K \) be the cone from Proposition 5.1, \( K = \{ z \in \Omega | d_\Omega(z, C') \leq \rho \} \).

Now assume \( x \in X_n \) for some \( n \). Since \( x \in X_n \) then \( f^n(x) \in X_0 \).

Now if \( f^n(x) \in K \), take \( f^{n+1}(x) \) which belongs to \( X_k \) for some \( k \) and begin the process again.

If \( f^n(x) \notin K \) then we examine its position. It lies outside the cone in between \( X_{qk} \) and \( X_{qk-2} \) for some \( k \). Since \( f^n(x) \in X_0 \), then it must be in \( G_k \). So we apply \( f^{qk-1} \) repeatedly to \( f^n(x) \) and examine its trajectory. There are two cases for the trajectory. Either the point enters the cone or the point does not enter the cone. If the point enters the cone \( K \) we apply \( f \) once more and begin the process again as in the previous paragraph.

So all we have left is to show that only a countable number of points never enter the cone under iteration. We will do this by showing for each \( k \) the set of points that never leave \( \overline{G_k} \) is at most one point.

Define \( S' \) to be the set of all points that remain forever in \( \overline{G_k} \) under iteration by \( f^{qk-1} \). Let \( S \) be the set of all points that remain forever in \( \overline{G_k} \) under iteration by \( f^{qk-1} \) and the well-defined inverse \( f^{-(qk-1)} \).

First examine \( S \). It is invariant under \( f^{qk-1} \), meaning \( f^{qk-1}(S) = S \). However since \( f^{qk-1} \) is expanding in the hyperbolic metric, then \( f^{qk-1} \) must strictly expand the diameter of the set \( S \) unless \( S \) has zero diameter. So \( S \) must have zero diameter, which implies either \( S \) consists of one point or \( S \) is empty.

We would like to show \( S' = S \). By definition \( S \subset S' \). Thus, all we need to show is that \( S' \subset S \). In order to achieve a contradiction, assume there is an \( x \in S' \) such that \( x \notin S \).

Call \( f^{qk-1} = F \) and consider the sequence \( \{ F^n(x) \} \). Since \( \overline{G_k} \) is compact, there is a convergent subsequence \( \{ F^n_j(x) \} \) converging to some \( x_0 \in \overline{G_k} \). The point \( x_0 \) must belong to
\[ F^m(x_0) = \lim_{j \to \infty} F^m(F^n_j(x)) = \lim_{j \to \infty} F^{m+n_j}(x), \]

which must belong \( S' \) since \( x \) and all its images belong to \( S' \). Furthermore, we can show \( x_0 \in S' \) since the above limits also hold if \( m \) is negative. Notice since \( x_0 \in S \) which is invariant and there is at most one point in \( S \) that \( x_0 \) must be a fixed point for \( F \). Since \( \{F^n_k(x)\} \) converges to \( x_0 \), for all \( \epsilon > 0 \) there is a \( n_0 \) such that if \( n_k > n_0 \) then \( d_{\Omega}(F^n_k(x), x_0) < \epsilon \). However, since \( F \) is expanding in the hyperbolic metric and \( x_0 \) is fixed under \( F \), \( d_{\Omega}(x, x_0) < \epsilon \) for all \( \epsilon > 0 \) and sufficiently large \( n \). So \( x = x_0 \in S \), which is a contradiction. Therefore \( S = S' \) and contains at most one point.

We have shown that at most one point remains in \( G_k \) under iteration. This implies that for each \( G_k \) all points, except perhaps one, enter the cone under iteration. Therefore, all but a countable number of points eventually map into the cone.

**Theorem 5.7.** For all but a countable number of limit points \( x \in J_\theta \), there is a sequence of positive integers \( n_k \), \( n_k \to \infty \) such that, for each \( n_k \) there is an embedded hyperbolic ball \( B_{n_k} \) centered at a point \( y_{n_k} \) such that for some ball \( B' \) and \( p \in C(0,1) \)

\[ f^{n_k+1} : (B_{n_k}, y_{n_k}) \to (B', p). \]

For the constant \( \rho \) from the cone \( K \) and a constant \( \kappa \) depending on \( \rho \), \( \frac{1}{K}||D_z(f^{n_k})(x)||_{\Omega} \leq \frac{\text{radius}_{\Omega}(B_{n_k})}{\rho} \leq \kappa||D_z(f^{n_k})(x)||_{\Omega} \), and \( d_{\Omega}(x, y_{n_k}) < 2\rho\kappa||D_z(f^{n_k})(x)||_{\Omega} \). The mapping \( f^{n_k} \) is univalent and has bounded distortion on \( B_{n_k} \).

**Proof.** Let \( x \in J \), a limit point, be given. Let \( K \) be the cone from Proposition 5.1. We will show whenever \( f^{n_k}(x) \in K \) for some \( n_k \) we get a mapping of hyperbolic balls. From the Proposition
we have an infinite number of these mappings with \( n_k \to \infty \) for all but a countable number of \( x \), so this will yield our result.

Remember that \( C' \) is the primary preimage of \( C(0,1) \). So for every \( y' \in C' \), any hyperbolic ball of finite radius in \( \Omega \) containing \( y' \) is disjoint from \( C(0,1) \) and \( f \) is univalent on the ball with \( y' \) mapping to some point \( p \in C(0,1) \).

Now if \( f^{nk}(x) \in K \) for some \( n_k \), we know there is a hyperbolic ball \( V' \) centered at a point \( y' \in C' \) which contains \( f^{nk}(x) \) and has \( \text{radius}_\Omega(V', y') = \rho \) and \( f : (V', y') \to (B', p) \) for some ball \( B' \) and \( p \in C(0,1) \).

Now examine \( f^{-nk} \) on \( V' \). The mapping \( f^{-nk} \) has bounded distortion on \( V' \), because \( f^{-nk} \) is univalent on the ball of radius \( 2\rho \) centered at \( y' \), which contains \( V' \). Call the distortion constant, \( \kappa \), for \( f^{-nk} \) on \( V' \). The distortion constant, \( \kappa \), depends only on \( \rho \). From the bounded distortion, we have \( f^{-nk}(V') \) contains a hyperbolic ball \( B \) centered at \( y \) with

\[
\frac{1}{\kappa} \|D_z(f^{-nk}(x))\|_\Omega \leq \frac{\text{radius}_\Omega(B)}{\rho} \leq \kappa \|D_z(f^{-nk}(x))\|_\Omega,
\]

so \( f^{nk+1} : (B, y_k) \to (B', p) \) as desired. Also the original ball \( V' \) contains \( f^{nk}(x) \), so \( d_\Omega(x, y_k) \leq 2\kappa \rho \|D_z(f^{-nk})'(x)\|_\Omega \).

Since from Proposition 5.6 we know that for all but a countable number of points, the point enters the cone infinitely many times and we can repeat this process whenever \( f^{nk}(x) \) lands in \( K \) for some \( n_k \), we are done.

\( \square \)

The Nearby Visits to \( C(0,1) \) Theorem will be our main tool for showing non-uniform porosity, as we will see in the next chapter.
Chapter 6

Non-uniform porosity

A compact set \( \Lambda \) is called *porous* (or *shallow*) if there exists an \( N \) such that for any \( z \in \Lambda \) and \( 0 < \epsilon < 1 \), given a square of any side length \( \epsilon \) centered at \( z \) and subdivided into \( N^2 \) squares of side length \( \epsilon/N \), then at least one of these squares is disjoint from \( \Lambda \). Equivalently a set \( \Lambda \) is porous if there exists a \( 0 < K < 1 \) such that for all \( z \in \Lambda \) and every \( 0 < r < 1 \), there is a ball \( B \) of radius \( s > Kr \), disjoint from \( \Lambda \) and contained in the ball of radius \( r \) about \( z \).

It can be shown directly that if a set \( \Lambda \) is porous then it must have upper box dimension less than 2 and hence have Hausdorff dimension less than 2.

This notion of porosity and the bounding of the upper box dimension using porosity depends heavily on having a uniform constant for all points \( z \in \Lambda \).

We will define a type of non-uniform porosity. A compact set \( \Lambda \) is called *non-uniformly porous* if for any \( z \in \Lambda \) there is a sequence of radii \( r_n \) with \( \lim_{n \to \infty} r_n = 0 \) and a constant \( 0 < K < 1 \) such that there is a ball \( B \) of radius \( s > Kr_n \), disjoint from \( \Lambda \) and contained in the ball of radius \( r_n \) about \( z \).

We can see from the definition that we have lost our uniformity and now cannot currently say anything about the upper box dimension.

**Theorem 6.1.** \( J_\theta \) is non-uniformly porous.

**Proof.** Let \( z \in J_\theta \), be given. We shall show \( J_\theta \) is non-uniformly porous at \( z \).

If \( z \in C(0,1) \) then \( J_\theta \) is non-uniformly porous at \( z \), since \( C(0,1) \) is in \( J_\theta \) and its interior is not in \( J_\theta \). Similarly if \( z \) is on a finite preimage of the circle, the same holds.

For \( z \) a limit point apply Theorem 5.7 to \( z \). Except for a countable number of limit points \( z \in J_\theta \), we have a sequence of positive integers \( n_k, n_k \to \infty \) such that, for each \( n_k \) there is an
embedded hyperbolic ball $B_{n_k}$ centered at a point $y_{n_k}$ such that for some ball $B'$ and $p \in C(0, 1)$,

$$f^{n_k+1} : (B_{n_k}, y_{n_k}) \to (B', p).$$

For the constant $\rho$ from the cone $K$ and a constant $\kappa$ depending on $\rho$

$$\frac{1}{\kappa} ||D_z(f^{-n_k})(z)||_\Omega \leq \frac{\text{radius}_\Omega(B_{n_k})}{\rho} \leq \kappa ||D_z(f^{-n_k})(z)||_\Omega,$$

and $d_\Omega(z, y_{n_k}) < 2\rho\kappa ||D_z(f^{-n_k})(z)||_\Omega$. The mapping $f^{n_k}$ is univalent and has bounded distortion on $B_{n_k}$.

So since from (McMullen, [Mc2], section 3.2) we know that $||D_z(f^{-n_k}(z))||_\Omega \to 0$, as $n_k \to \infty$. This means the hyperbolic radii of the balls and $d_\Omega(z, y_k)$ go to zero as $k$ approaches infinity. So similarly the Euclidean radius and distance do as well, due to the comparison between the hyperbolic and Euclidean metrics.
We will construct the sequence of radii in the following manner. We have a sequence of hyperbolic balls $B_{n_k}$. Begin with some radius $r_0$, such that the ball $B_{r_0}(z)$ of Euclidean radius $r_0$, contains $B_{n_0}$. From the Theorem 5.7, we know $f^{n_0+1}(B_{n_0})$ maps to some $(B', p)$ such that $p \in C(0,1)$. We take a ball $U$ inside $D(0,1) \cap B'$ and consider $f^{-(n_0+1)}(U)$; it contains a ball of radius $cd(x, y_{n_0})$ for some $c$ and is disjoint from $J_\theta$. We can then take a radius $r_1 < r_0$ such that $B_{n_1}$ is contained inside $B_{r_1}(z)$ and there is a ball of radius $cd(x, y_{n_1}$ disjoint from $J_\theta$. We repeat this process and we will have achieved a sequence of radii and a sequence of disjoint balls, all taking the same fixed proportion of $B_{r_n}(z)$.

If $z$ is one of the countable number of limit points that are not covered by Theorem 5.7, then we know $z$ is a either a repelling periodic point or a preimage of a repelling periodic point. If $z$ is fixed point, we take a ball of some radius $r$ around $z$, $B_r(z)$, which is small enough that it maps into itself under $f^{-1}$. Since the Fatou set is non-empty $J_\theta$ must be nowhere dense. Thus there is a ball $U$ contained in $B_r(z)$ such that $U \cap J_\theta = \emptyset$. Under $f^{-1}$, $(B_r(z))$ will contain a ball of some smaller radius and the preimage of $U$ will contain a ball that takes up the same proportion of the ball. We can continue this process indefinitely, yielding the sequence of radii as desired. If the point $z$ is periodic with period $k$, the same construction holds as well using $f^{-k}$.

If $z$ is a preimage of a periodic point, we can pull back the sequence of radii and balls disjoint from $J_\theta$ from the period point on the forward orbit of $z$.

Non-uniform porosity implies Lebesgue measure zero. This fact follows easily from the fact that a point at which there is non-uniform porosity cannot be a Lebesgue density point. We can now see directly the following corollary.

**Corollary 6.2.** The Lebesgue measure of $J_\theta$, for $\theta$ an irrational number is zero.
Chapter 7

Conclusions

Now that we have shown that $J_\theta$ is non-uniformly porous, many questions remain. Is $J_\theta$ actually porous? While theoretically possible, it is unlikely that the methods used in this paper could be adapted to show porosity. The only possible way to achieving such an adaption would be to be able to pick a cone so that all points of the primary limb $X_0$ lie inside it, which seems unlikely.

However, despite that porosity seems out of reach. It appears likely that $J_\theta$ has Hausdorff dimension less than two. There are two possible ways of showing this given our results. The first would be to show that non-uniform porosity itself leads to a dimension bound. The second is to show that non-uniform porosity is equivalent to mean porosity or weak mean porosity.

One other item of note about $J_\theta$ is about the immediate basin of infinity itself. It appears to not be a John domain. A set $\Lambda$ in $\hat{\mathbb{C}}$ is a John domain if there is a center $z_0$ and an $\epsilon > 0$ such that for all $z_1 \in \Lambda$ there exists an arc $\gamma \subset \Lambda$ connecting $z_0$ and $z_1$ and $d(z, \partial \Lambda) > \epsilon |z - z_1|$ for all $z \in \gamma$. If $\Lambda$ is unbounded and its boundary is compact then $z_0 = \infty$. For $J_\theta$ all we would need to show in order to prove that the immediate basin of infinity is not a John domain, would be to show that the set $S$ for points that did not enter the cone $K$ under iteration was non-empty for an infinite number of $n$. This would provide the fact that if we picked $z_1$ close to the critical point 1, we would not be able to find an appropriate path $\gamma$.

Another interesting question is what can be said about the full Julia set $J_{f_\theta}$. Of the results proven here, the finding of the cone $K$, Proposition 5.1, is certainly true for $J_{f_\theta}$. This comes from the fact that the filled in Julia set $K_{f_\theta}$ has $J_\theta$ as its boundary. The Corollary 5.7 is
also true for $J_{f\theta}$ as well. These might be of some use, when attempting to find or estimate the measure of $J_{f\theta}$ which is unknown.
References


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