Christiaan Huygens and the Problem of the Hanging Chain

John Bukowski

John Bukowski (bukowski@juniata.edu) is Associate Professor and Chair of the Department of Mathematics at Juniata College in Huntingdon, Pennsylvania. He received B.S. degrees in mathematics and physics from Carnegie Mellon University and his Ph.D. in applied mathematics from Brown University. He currently serves as Governor of the Allegheny Mountain Section of the MAA. He was a 1998–1999 Project NExT Fellow (silver dot) and is now Co-Coordinator of his Section NExT program. He finds time to do mathematics when he is not busy as College Organist, choir accompanist, church organist, or piano soloist. One of his accomplishments at Juniata is his performance of a Mozart piano concerto in 2006. He and his wife Cathy Stenson (also a mathematician) have two sons, David and Daniel, who like to ask them mathematical questions.

In his *Discorsi* [**10**] of 1638, Galileo wrote much about strength of materials and cross-sections of beams, and the parabola kept appearing in these contexts. After correctly instructing the reader how to draw such a curve via projectile motion, Galileo then explained, "The other method of drawing the desired curve. . . is the following: Drive two nails into a wall at a convenient height and at the same level. . . . Over these two nails hang a light chain. . . . This chain will assume the form of a parabola. . . ." If we follow Galileo's instructions and hang such a chain, the resulting curve certainly looks like it could be a parabola. In fact, Galileo was merely stating what was commonly thought about the problem of the hanging chain at the time, as it was widely accepted in the early seventeenth century that a hanging chain did indeed take the form of a parabola. We shall see that the young Christiaan Huygens was the first person to prove that such a chain did not hang as a parabola. It was then not until much later in the century that mathematicians finally understood that the chain hung in the form of a catenary, as we know today.

Early history of the problem

The problem of the hanging chain is thought to have first appeared about a century earlier, when Leonardo da Vinci sketched a few hanging chains in his notebooks [**23**, p. 21]. Other prominent seventeenth century mathematicians also considered the problem. In 1614, the Flemish mathematician Isaac Beeckman (1588–1637) wrote in his notebook, "Let there hang from a beam a cord. . . attached at the ends so that it hangs loosely and freely," while including an accompanying sketch [**2**]. Beeckman also asked his friend René Descartes (1596–1650) about it. In the writings of Descartes, there is a note [**6**], "Sent to me by Isaacus Middelburgensis [Beeckman] whether a cord. . . affixed by nails. . . may describe part of a conic section." There is no indication that Descartes attempted to solve the problem or even that he discussed it further with Beeckman.

This problem also appeared in the works of the famous Dutch mathematician Simon Stevin (1548–1620), whose *Les Oeuvres Mathématiques* was published posthumously in 1634, with annotations by the French mathematician Albert Girard (1595–1632). In his writings on statics, Stevin considered problems dealing with ropes [**21**, p. 508]. Here Girard wrote, "…because the other slack or taut ropes are parabolic lines (as I have in the past proven around the year 1617), as I will prove after this at the end of the following corollary…." It appears that there was no proof from the year 1617, however, as Girard continues after Stevin's corollary, "To satisfy my last promise which precedes the last corollary, and not having the spare time however to place here a copy of my entire proof, I will give it another time in public, with my other works, in return for the help of God, when the research of the sciences will be very commendable, which it is not at present." Thus the work ends without an appearance of a proof from Girard, who was also dead at the time of the publication of Stevin's *Oeuvres*.

The young Christiaan Huygens

Christiaan Huygens (1629–1695) was the second son of Constantijn Huygens and Susanna van Baerle in an important Dutch family. Constantijn, well-known as a poet and a composer, worked in the service of the government and was somewhat wealthy as a result. He had important intellectual contacts, and he was able to afford private academic instruction for his children. In 1644, Constantijn hired Jan Jansz. Stampioen de Jonge as a mathematics tutor for Christiaan and his older brother, also named Constantijn. It is through his tutor that Christiaan Huygens may have learned of the problem of the hanging chain, as Stampioen recommended the works of Stevin to him in a letter in 1645 [**12**, pp. 6–7]. In addition, his father had been acquainted with Girard.

Christiaan Huygens's father also exchanged correspondence with the French monk Marin Mersenne (1588–1648). Mersenne is often referred to as a clearinghouse for mathematical and scientific ideas, as he corresponded with many of the best thinkers throughout Europe. As a proud father, Constantijn Huygens was pleased to tell Mersenne about his brilliant seventeen-year-old son Christiaan. Consequently, Mersenne sent a letter to the young Christiaan on October 13, 1646, thereby beginning a series of correspondence that would last until Mersenne's death in September 1648. In these letters, Huygens proved to Mersenne that the hanging chain did not take the form of a parabola, as had been claimed by Galileo eight years earlier. Of course Huygens was correct, as we know now that the curve formed by such a chain is a catenary, or a hyperbolic cosine. The term "catenary" is the English version of the Latin term *catenaria*, originally coined by Huygens himself in a November 1690 letter to Gottfried Leibniz [**13**, p. 537]. It is interesting to note that these words all derive from the Latin *catena*, meaning "chain." So we say today that the shape of a hanging chain is…a chain!

The correspondence between Huygens and Mersenne

Huygens concluded his first letter to Mersenne on October 28, 1646, with the promise, "I will send you in another letter a proof that a hanging cord or chain does not make a parabola, and what should be the pressure on the mathematical cord or one without gravity to make one; I have found such a proof not long ago." [**12**, p. 28] Since the hanging chain itself is not a parabola, Huygens states here that he has figured out how to make the chain into the shape of a parabola with the help of some external forces.

Intrigued by Huygens's claim about the hanging chain, Mersenne responded on November 16 wishing to see the proof "that the chain, or stretched cord, hanging under its own weight in the middle, does not make a parabola, as put forth by Galileo. . . ." [**12**, p. 31] He also states his desire to see the additional proof of Huygens, while taking it one step further, as he tells Huygens that he wishes to see what "must be the pressure for it to make the aforementioned parabola, and if you add, as it is necessary, the pressure for it to make a hyperbola and an ellipse, then you will surpass yourself." Later that month, Huygens responds to Mersenne with a complete proof of his claims concerning the hanging chain.

Huygens's proof that the hanging chain is not a parabola

Huygens begins his proof [**12**, pp. 34–40] by stating four assumptions. First he writes, "I suppose therefore first that the whole cord depends only on some gravity, tending toward the center of the earth, one parallel to another." He continues, "Secondly, that two or more weights... attached to the cord... which is held at [the endpoints], can remain at rest in only one way." The third and fourth assumptions deal with the way in which the cord hangs, that it hangs in the same way even when one's hand holds the chain at certain points.

Huygens then begins to set up his argument with the following proposition and the corresponding Figure 1.

Proposition 5. *If there are so many weights that one wants S, R, P, Q hanging from a cord ABCD, I say that MD and BC continued intersect at L on the hanging diameter of the weights P and Q. AB and DC [intersect] at K on the hanging diameter of the weights R and P, and in this way the rest. . . .*

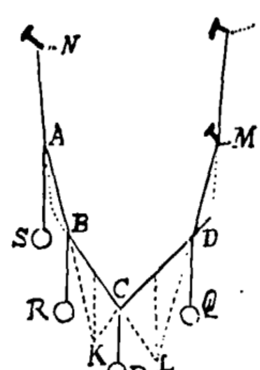

Figure 1. Huygens's sketch to accompany Proposition 5.

In Proposition 5, Huygens hangs equal weights from a weightless chain, and he discusses the positions in which these weights hang, such that extensions of certain segments intersect on the "hanging diameter of the weights" between them. This term "hanging diameter of the weights" refers to the vertical line halfway between the two (equal) weights. This proposition is based on a similar result given by Stevin [**21**, pp. 454–455], in which he refers to this vertical line as the "perpendicular of the weight."

It is interesting to note that at this point in his discussion, Huygens switches from French to Latin, possibly realizing that he is getting into some serious mathematics! In his Proposition 6 that follows, Huygens removes the hanging weights and instead uses a chain composed of segments of equal weight, as we see in Figure 2. It is this chain that Huygens uses for the rest of his proof.

Proposition 6. *By the same method it is shown that if AB, BC, CD, etc., are lines of equal weight, and two extensions such as CD and FE intersect each other in the hanging diameter of the weight, it is in the middle of DE. Therefore, if all of the weights are equal, DR must be equal to RE, EP equal to PF, etc.*

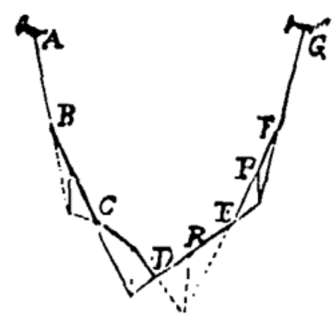

Figure 2. Huygens's sketch to accompany Proposition 6.

The resulting shape of the weighted chain here is identical to that of the cord in Figure 1. Huygens notes that the extensions of the segments intersect in the same way as before, on the hanging diameter of the weight. He adds that this vertical line bisects the segment directly above the intersection, a crucial piece of information for his Proposition 8 to come.

Now that Huygens has set up his chain and described the geometry of its position, he continues with Proposition 7, awkwardly stated and argued, in which he explains that the chain must in fact hang as described in the two previous propositions and not in any other way, concluding, "I say this is to be the position in which they are able to hang, and they must."

It is instructive to note that Huygens's education included the study of geometrical works such as Euclid's *Elements*, Apollonius' *Conics*, and the writings of Descartes, so it is not surprising that the young Huygens approached the problem of the hanging chain in this way. Descartes once wrote in a letter [**7**] to Princess Elizabeth, "In the solution of a geometrical problem. . . I use no theorems except those which assert that *the sides of similar triangles are proportional*, and that in a right triangle the square of the hypotenuse is equal to the sum of the squares of the sides." We now see that the upcoming argument—the heart of Huygens's proof, in fact—uses similar triangles in an essential way.

Proposition 8. *Let the hanging chain HGABCDK consist of lines of equal length, weight, and shape; I say the points of connection GABCDK cannot coincide on the same parabolic line.*

Huygens begins his proof of Proposition 8 in the following way:

From Proposition 6 it is evident in what manner these lines may be bound to hang truly, so that H may be in the middle of BC, P in the middle of CD, etc. And so now the parabola $RABCF$ having been described, passing through three points A, B, C, I say that this is not to pass through the point D and the rest. . . .

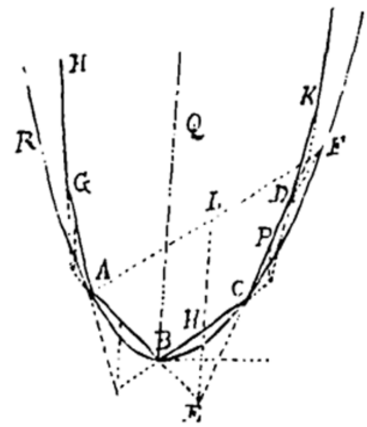

Figure 3. Huygens's sketch to accompany Proposition 8.

Here, Huygens sets up a hanging chain of equal segments and the parabola $RABCF$ in the same diagram, as shown in Figure 3. He relies on Proposition 6 to say that the extensions of the segments intersect on the hanging diameters of the weights. He then claims that the parabola and the chain intersect only at the points A, B, C, (since one can always fit a parabola to any three points) and nowhere else. He plans to show that the parabola $RABCF$ does not pass through the other points H, G, D, and K, on the chain. Once Huygens proves this claim, he will have shown that the chain and the parabola are indeed different shapes—the punch line to this whole discussion! To that end, Huygens continues with the following:

. . . for ECD may be extended until it may be that $\frac{FC}{CE} = \frac{AB}{BE}$, and then AF may be drawn, and this therefore will be parallel to BC and similarly will be divided in two by the line EL at L, therefore the point F will be on the same parabola with the points A, B, C, for EL is a diameter of the parabola B, and not the point D.

Huygens refers here to the definition of diameter put forth by Apollonius in the third century B.C., that a diameter is a line that bisects any set of parallel segments across a parabola. We note that a diameter of a parabola is always parallel to the axis of the parabola. In this part of the argument, Huygens creates his first pair of similar triangles, $\triangle FEA$ and $\triangle CEB$, and he equates the ratios of corresponding sides. He claims that the point F is on the given parabola, but that the point D is not. He then explains why this is the case, in the final sentence of the paragraph:

For otherwise the line $ECDF$ might have been obliged to cut the parabola in three points, which is absurd, or the point D to coincide with the point F, which is impossible, for $FC > AB$ or DC, since $CE > BE$.

Of course we cannot draw a line that intersects a parabola in three points, so this is not an option. The other option is that D and F are really the same point, in which case the line would only intersect the parabola in two points. But Huygens uses the above equation, $\frac{FC}{CE} = \frac{AB}{BE}$, to show that this is not possible. From Figure 3, we see immediately that $CE > BE$, and so $FC > AB$. But $AB = DC$, since all segments of the chain are equal in length. Therefore, Huygens shows that $FC > DC$, meaning that F and D are indeed distinct points—where F is on the parabola and D is not. Proposition 8 continues with similar arguments to show that the other points of the chain are not on the parabola either. In this way, Huygens shows that the chain does not hang in the form of a parabola.

To make sure that we are absolutely certain about the result obtained in Proposition 8, Huygens gives his next proposition:

Proposition 9. *And thus from whence it may be clear, if the hanging chain having been composed of lines of the same length and weight, that the parabola which is described by its extremum [vertex]... and by two of its other points... , through none of its other points to pass, but to exclude them all which are below the given points... , [and] certainly those which are above them. Therefore no chain hangs according to a parabolic line.*

Forcing the chain to hang as a parabola

Huygens then goes on to consider the additional problem he mentioned to Mersenne, that of actually forcing the chain to somehow hang in the shape of a parabola. As we see in Figure 4, the idea of his Proposition 11 is to hang equal weights from a weightless string at equal intervals along the horizontal (as opposed to equal intervals along the string itself, which is what he did earlier in Proposition 5). Huygens says then

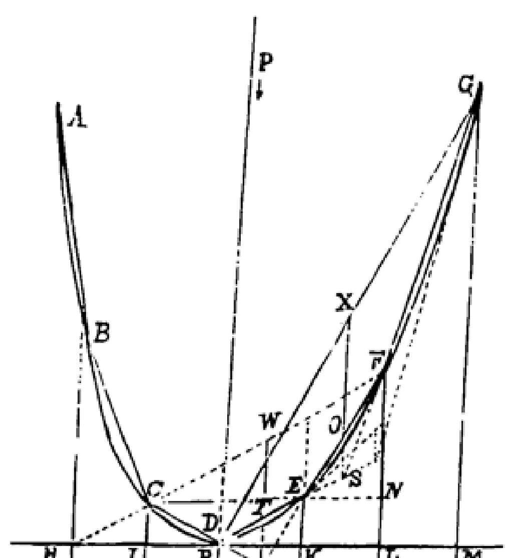

Figure 4. Huygens's sketch to accompany Proposition 11.

that B, C, D, E, F, and G are points on the same parabola. He proceeds to mention that the extensions of certain segments intersect on the hanging diameters of the weights, as we see at points Q and S in the figure. Having established that *ABCDEFG* is indeed a parabola, he concludes this proposition by stating what is known numerically about parabolas, "*KE* is known to be 1, *LF* 4, *MG* 9 and thus in succession." Huygens is indeed correct that this arrangement of weights leads to a parabola, and this is in fact the basic principle behind the parabolic shape of the cables of a suspension bridge. See [**22**, p. 522 (Problem 39)] for the setup and a brief discussion of this problem.

Clearly excited by what he has written about the parabola in Proposition 11, Huygens goes on in a Manifestum to propose a second way to make a parabola, as shown in Figure 5: "Hence it is clear that if on the string... might be placed little beams or parallelepipeds of equal weight, size, and shape, the points A, B, C, etc., press on the string, each one to be on the same parabola...." Since it is apparently so clear, Huygens offers no proof of this statement, although the idea seems reasonable enough, as does the figure he draws. Twenty-two years later, however, in the margin of another version [**15**, p. 43] of this same argument, Huygens writes, "*non sequitur neque est verum*"—"it does not follow, nor is it true." Here in 1668, Huygens finally realizes his error in this statement from his youthful correspondence. In fact, if one stacks such rectangles inside a hanging cord, an arc of a circle will result, as Truesdell [**23**, p. 46] briefly discusses. In the case of the weights hanging from the string (as in Proposition 11), the cord is pulled downward, whereas here the rectangles push the cord outward in a normal direction (certainly not downward!), causing the two situations to produce different shapes.

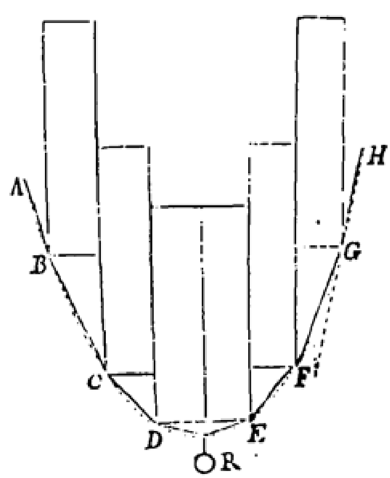

Figure 5. Huygens's sketch to accompany the Manifestum.

Later history of the hanging chain

After Huygens's letters to Mersenne in 1646, the problem of the hanging chain was not studied for many years. The next known investigation of the problem was by Joachim Jungius (1587–1657), who is said to have also shown that the hanging chain is not a parabola, in a work that appeared in 1669. A better-known attempt to understand the problem was by the Jesuit Father Ignace Gaston Pardies (1636–1673), who considered it in *La Statique, ou la Science des Forces Mouvantes* [**20**], published in 1673. In this work, Pardies also proves that the hanging chain does not take the shape of a parabola.

In contrast to the complicated geometrical arguments of Huygens, the proof by Pardies is much more elegant. Here, Pardies uses his own continuous analogue of Stevin's theorem in his proof, in which he states that two tangents to the chain intersect on the vertical line through the center of gravity of the chain [**23**, p. 51].

We notice that although Huygens, Jungius, and Pardies were able to show that the hanging chain is not a parabola, they were not able to say what shape the hanging chain actually is. More years passed before Jacob Bernoulli posed the problem in *Acta Eruditorum* in May 1690, challenging readers to come up with the solution of the actual shape of the hanging chain [**3**]. Just over a year later, in the June 1691 *Acta*, three solutions to the problem of the hanging chain appeared—those of Leibniz, Johann Bernoulli, and Christiaan Huygens!

Summary of Huygens's 1691 paper in *Acta Eruditorum*

Huygens's paper in *Acta Eruditorum* [**16**], although less than two pages in length, was filled with many facts about the catenary. Although Huygens did not give a formula for the catenary curve, his list of results clearly indicates that he knew a great deal about this curve, shown in Figure 6. Referring to the figure, Huygens sets up the problem in the following way:

> If the chain *CVA* may be suspended by strings *FC*, *EA* attached at both ends... so that the ends *C* and *A* may be of equal height, and so that the angle of inclination *CGA* of the extended strings might be made, and the position of the whole chain, whose vertex is *V* and axis is *VB*.

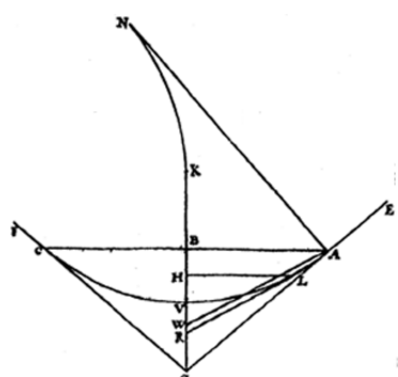

Figure 6. Huygens's sketch of the catenary from his 1691 paper.

In this arrangement, the segment *EA* is extended to the point *G*, such that the segment *AG* is tangent to the curve at *A*. Similarly, *CG* is tangent to the curve at *C*. In this public presentation of his results, Huygens analyzes the curve in different ways, considering only the following special cases: (1) $\triangle GBA$ is a 3-4-5 right triangle, (2) $\angle CGA$ is a 60° angle, and (3) $\angle CGA$ is a right angle. We discuss here a few of Huygens's results that illustrate his precise understanding of the curve *CVA*.

One of the characteristics of the curve that interests Huygens is the arclength *VA*, which is one-half the length of the chain. He states that for any given angle *CGA*, "the ratio of the axis *BV* to the curve *VA* will be given." Specifically he mentions, "If the sides GB, BA, AG are as 3, 4, 5, the curve VA will be triple the axis VB," while offering no justification for the results.

Next, Huygens discusses the radius of curvature at the vertex V, which he defines as "the semidiameter of the largest circle that may be drawn through this vertex which falls entirely inside the curve." He says that if $\angle CGA$ is a 60° angle, the radius of curvature is the axis BV. Additionally, he states that if $\angle CGA$ is a right angle, the radius of curvature equals the arclength VA.

Huygens also discusses the curve in the context of his own earlier theory of evolutes, which he developed in 1659 as part of his work on the pendulum clock [**17**]. He states that "the curve KN, by the unrolling of which, together with the straight line KV, the radius of curvature at the vertex, the curve VA is described." In modern terminology, the curve NKV is the evolute, and the catenary VA is the involute. See [**26**] for a thorough discussion of Huygens's work with evolutes.

The final result of this paper in the *Acta* is numerically impressive, accurate to five decimal places. Huygens states that "if the angle CGA may be right, and the axis BV may be specified to be of 10000 parts; BA will be 21279, not one less. Also the curve VA... is shown to be of 24142 parts, not one less."

Huygens gives virtually no details about how any of these results were obtained. The editors of his *Oeuvres Complètes*, however, point the reader to a collection of notes [**14**] where Huygens derives several formulas that he uses to obtain some of these results. It is most interesting to consider that Huygens worked on this problem of the hanging chain at the beginning of his (mathematical) life in 1646 and again near the end of his life. When presented with the challenge of the hanging chain forty-five years after his first encounter with it, Huygens still used a great deal of classical geometry to obtain his results, although the two approaches are quite different.

Concluding remarks

The catenary problem is still of interest today. It appears in calculus textbooks (see, for example, [**22**, pp. 557–560, 567–568]), where the forces on a section of chain are balanced to derive the differential equation

$$\frac{d^2y}{dx^2} = \frac{1}{a}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

When combined with the conditions $\dfrac{dy}{dx} = 0$ and $y = a$ when $x = 0$, one obtains the solution $y = a\cosh\left(\dfrac{x}{a}\right)$. The problem is also a standard one in studies of the calculus of variations [**9**]. One can minimize the potential energy of the chain by minimizing the integral $\displaystyle\int_A^B y\sqrt{1 + (y')^2}\,dx$, where A and B are the endpoints of the chain. By using the Euler characteristic equation and an appropriate choice of constants, one gets the hyperbolic cosine solution obtained above.

For readers who are intrigued by this curve and would like to learn more, some web sites contain interactive demonstrations where one can explore the catenary curve graphically [**18**], [**19**]. There is also a clever article detailing the construction of catenaries and parabolas with paper clips [**24**]. The catenary has also made appearances in MAA journals [**25**], [**1**], including recent ones in the *CMJ* [**5**], [**11**]. One can even listen to Keith Devlin talking about the difference between the catenary and the parabola on National Public Radio [**8**]!

For those wishing to learn more about Christiaan Huygens, a great place to start is with the works of Henk Bos, specifically a delightful article on his life and works [**4**].

Translation Note. All translations of Huygens, Girard, and Beeckman (from Latin, French, and Dutch) are those of the author, unless otherwise specified.

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