The Numerical Range of a Matrix

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January 13, 2015
**Definition (Numerical Range)**

Let $A$ be an $n \times n$ matrix with entries in $\mathbb{C}$. Then the numerical range of $A$ is given by

$$W(A) = \{\langle Ax, x \rangle : x \in \mathbb{C}^n, \|x\| = 1\} = \{x^* Ax : x \in \mathbb{C}^n, \|x\| = 1\}.$$
Some Examples

\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & i & 0.5 \\ 0 & 0 & -i \end{bmatrix} \]

\[ B = \begin{bmatrix} -1 & 2 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \]
Basic Properties

The numerical range of a matrix, $W(A)$ is a compact and convex subset of $\mathbb{C}$.

![Diagram 1](image1.png)

![Diagram 2](image2.png)
The numerical range of a matrix, $W(A)$ is a compact and convex subset of $\mathbb{C}$.

$W(A)$ also contains the eigenvalues of $A$.

Proof: If $\lambda$ is an eigenvalue of $A$, then we pick a corresponding unit eigenvector, $x$. Then $\langle Ax, x \rangle = \langle \lambda x, x \rangle = \lambda \langle x, x \rangle = \lambda$. 
A matrix $U$ is unitary if $U^*U = I$ or $U^{-1} = U^*$.

If $U$ is a unitary $n \times n$ matrix, $W(U^*AU) = W(A)$ for any $n \times n$ matrix $A$. We say $W(A)$ is invariant under unitary similarities.

If $A$ is unitarily reducible, that is, unitarily similar to the direct sum $A_1 \oplus A_2$, then $W(A)$ is the convex hull of $W(A_1) \cup W(A_2)$.

$$A_1 \oplus A_2 = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$
Unitarily Reducible Example

\[
A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & i \\ 1 & 3 \end{bmatrix}, \quad A_1 \oplus A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & i \\ 0 & 0 & 1 & 3 \end{bmatrix}
\]
A square matrix $A$ is *normal* if $A^* A = AA^*$. A matrix is normal if and only if it is unitarily similar to a diagonal matrix. In this case, $A$ is unitarily reducible to a direct sum of $1 \times 1$ matrices which are its eigenvalues, so $W(A)$ is the convex hull of its eigenvalues.

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 - i & 0 & 0 & 0 \\ 0 & 0 & 1 + i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
Characterizing Shapes: $2 \times 2$ matrices

If $A$ is an irreducible $2 \times 2$ matrix, $W(A)$ is an ellipse with foci at the eigenvalues.

$$A = \begin{bmatrix} 0 & 3 \\ -1 & 4 \end{bmatrix}$$
Characterizing Shapes: $3 \times 3$ matrices
A doubly stochastic matrix is one whose real number entries are non-negative and each row and column sums to 1. For example:

\[
\begin{bmatrix}
\frac{2}{5} & 0 & \frac{2}{5} & \frac{1}{5} \\
\frac{2}{5} & 0 & \frac{1}{5} & \frac{3}{5} \\
0 & \frac{3}{5} & 0 & \frac{2}{5} \\
\frac{1}{5} & \frac{2}{5} & \frac{2}{5} & 0
\end{bmatrix}
\]

Key fact: Every doubly stochastic matrix is unitarily reducible to the direct sum of the matrix [1] with another matrix:

\[U^* AU = [1] \oplus A_1,\]

where \(U\) is a unitary matrix with real entries.
Given the characterization of shapes of the numerical range for $3 \times 3$ matrices and unitary reducibility $U^T A U = [1] \oplus A_1$, there are three possibilities:

- $W(A_1)$ is the convex hull of a point and an ellipse (with the point lying either inside or outside the ellipse);
- the boundary of $W(A_1)$ contains a flat portion, with the rest of it lying on a 4th degree algebraic curve;
- $W(A_1)$ has an ovular shape, bounded by a 6th degree algebraic curve.

All possibilities occur in these three categories and we can characterize which occurs.
Theorem

Let $A$ be a $4 \times 4$ doubly stochastic matrix. Then the boundary of $W(A)$ consists of elliptical arcs and line segments if and only if

$$\mu := \text{tr} A - 1 + \frac{\text{tr} A^3 - \text{tr}(A^T A^2)}{\text{tr}(A^T A) - \text{tr} A^2}$$

is an eigenvalue of $A$ (multiple, if $\mu = 1$). If, in addition,

$$\alpha = \text{tr} A - 1 - 3\mu > 0, \beta = (\text{tr} A - 1 - 3\mu)^2 - \text{tr}(A^T A) + 1 + 2(\det A) / \mu + \mu^2 > 0,$$

then $W(A)$ also has corner points at $\mu$ and $1$, and thus four flat portions on the boundary. Otherwise, $1$ is the only corner point of $W(A)$; and $\partial W(A)$ consists of two flat portions and one elliptical arc.
$4 \times 4$ D-S examples: ellipses and line segments

$\alpha = 7/24, \beta = -59/576$

$\alpha \approx 0.65, \beta \approx 0.31$

$\alpha = 43/96, \beta = -779/9216$

$\alpha \approx 0.77, \beta \approx 0.21$
4 × 4 D-S Examples: 4th degree curves and flat portions

4th degree curves and 1 with flat portion on the left

4th degree curve and 1 with flat portion on the right
$4 \times 4$ D-S Examples: Ovular
Let $A_\phi = Ae^{i\phi}$.

- $W(A)$ vs. $W(A_\phi)$
Graphing the boundary of the numerical range

Let $A_\phi = Ae^{i\phi}$.

- $W(A)$ vs. $W(A_\phi)$
- $H_\phi := \frac{A_\phi + A^*_\phi}{2}$, $K_\phi := \frac{A_\phi - A^*_\phi}{2i}$
- $W(H_\phi) = \text{Re}(W(A_\phi)) = [\lambda_{\text{min}}, \lambda_{\text{max}}]$.
- $H_\phi \nu = \lambda_{\text{max}} \nu$.
  $\lambda_{\text{max}} = \langle H_\phi \nu, \nu \rangle = \text{Re} \langle A \nu, \nu \rangle$.
- $\langle A \nu, \nu \rangle \in W(A)$.
- Singularity: $W(A_\phi)$ has a vertical flat portion $\Rightarrow \lambda_{\text{max}}$ has multiplicity 2
Graphing the boundary of the numerical range

Let $A_{\phi} = Ae^{i\phi}$.

- $W(A)$ vs. $W(A_{\phi})$
- $H_{\phi} := \frac{A_{\phi} + A_{\phi}^*}{2}$, $K_{\phi} := \frac{A_{\phi} - A_{\phi}^*}{2i}$
- $W(H_{\phi}) = \text{Re}(W(A_{\phi})) = [\lambda_{\text{min}}, \lambda_{\text{max}}]$.
- $H_{\phi}v = \lambda_{\text{max}}v$.
  $\lambda_{\text{max}} = \langle H_{\phi}v, v \rangle = \text{Re}\langle Av, v \rangle$.
- $\langle Av, v \rangle \in W(A)$.
- Singularity: $W(A_{\phi})$ has a vertical flat portion $\Rightarrow \lambda_{\text{max}}$ has multiplicity 2

Alternative (Kippenhahn, 1951): Let $F(x : y : t) := \det(H_0x + K_0y + lt)$. Then $W(A)$ is the convex hull of the dual curve to $F(x : y : t) = 0$. 
The Gau-Wu number

**Definition**

*(Gau-Wu number, 2013)*

\[ k(A) = \max \# \{x_1, \ldots, x_k \} \]

\[ \forall j, \langle Ax_j, x_j \rangle \in \partial W(A) \]

\{x_1, \ldots, x_k\} orthonormal

\{x_1, \ldots, x_k\} \subset \mathbb{C}^n

**Basic results:**

- \( 1 \leq k(A) \leq n \)
- Points on parallel support lines of \( W(A) \) come from orthogonal vectors.
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- $1 \leq k(A) \leq n$
- $k(A) \geq 2$ if $n \geq 2$

Figure: $A \in M_2(\mathbb{C})$. $k(A) = 2$
Basic examples

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**Figure:** \( A \in M_2(\mathbb{C}). \ k(A) = 2 \)

- Vertical flat portion \( \ell_1 \Rightarrow \) pair of orthogonal eigenvectors \( u, v \) of \( H_0 \), with \( \langle Bu, u \rangle, \langle Bv, v \rangle \in \ell_1 \cap \partial W(B) \).

**Figure:** \( B \in M_3(\mathbb{C}) \)
Basic examples

Basic results:

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Figure: $A \in M_2(\mathbb{C})$. $k(A) = 2$

- Vertical flat portion $\ell_1 \Rightarrow$ pair of orthogonal eigenvectors $u, v$ of $H_0$, with $\langle Bu, u \rangle, \langle Bv, v \rangle \in \ell_1 \cap \partial W(B)$.
- Let $\ell_2$ be a parallel support line, and let $\langle Aw, w \rangle \in \ell_2 \cap \partial W(B)$. Then $w \perp u, w \perp v$.
Basic examples

Basic results:
- \(1 \leq k(A) \leq n\)
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Figure: \(A \in M_2(\mathbb{C}). k(A) = 2\)

- Vertical flat portion \(\ell_1 \Rightarrow\) pair of orthogonal eigenvectors \(u, v\) of \(H_0\), with \(\langle Bu, u \rangle, \langle Bv, v \rangle \in \ell_1 \cap \partial W(B)\).
- Let \(\ell_2\) be a parallel support line, and let \(\langle Aw, w \rangle \in \ell_2 \cap \partial W(B)\). Then \(w \perp u, w \perp v\).
- Thus \(k(B) = 3\).
Example 1 of an irreducible $4 \times 4$ matrix

Let $A \in M_4(\mathbb{C})$ irreducible, with $F(x : y : t) = 0$ a curve having two nodes.

Dual curve:

- Vertical flat portion $\ell_1 \Rightarrow$ pair of orthogonal eigenvectors $u, v$ of $H_0$, with $\langle Au, u \rangle, \langle Av, v \rangle \in \ell_1 \cap \partial W(A)$.
- Let $\ell_2$ be a parallel support line, and let $\langle Aw, w \rangle \in \ell_2 \cap \partial W(A)$. Then $w \perp u$, $w \perp v$. 
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- Let $\ell_2$ be a parallel support line, and let $\langle Aw, w \rangle \in \ell_2 \cap \partial W(A)$. Then $w \perp u, w \perp v$.
- Thus $k(A) = 4$. 
Example 2 of an irreducible $4 \times 4$ matrix

Let $A \in M_4(\mathbb{C})$ irreducible, with $F(x : y : t) = 0$ a curve having three nodes. Dual curve:
Example 2 of an irreducible $4 \times 4$ matrix

Let $A \in M_4(\mathbb{C})$ irreducible, with $F(x : y : t) = 0$ a curve having three nodes. Dual curve:

Lemma

(C,R,S,S, 2015) Let $A \in M_4(\mathbb{C})$, $H_\phi := (A_\phi + A_\phi^*)/2$. $S = \{\phi : H_\phi$ has a maximum e-value of multiplicity $\geq 2$. Then:

1. $\forall \phi \in S$, $H_\phi$ has exactly three distinct eigenvalues $\Rightarrow k(A) < 4$. 

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Example 2 of an irreducible $4 \times 4$ matrix

Let $A \in M_4(\mathbb{C})$ irreducible, with $F(x : y : t) = 0$ a curve having three nodes.

Dual curve:

Lemma

$S = \{\phi : H_\phi \text{ has a maximum e-value of multiplicity } \geq 2\}$. Then:

1. $\forall \phi \in S, H_\phi \text{ has exactly three distinct eigenvalues } \Rightarrow k(A) < 4.$
2. $S \neq \emptyset \Rightarrow k(A) > 2.$
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2. $S \neq \emptyset \Rightarrow k(A) > 2$.

Thus $k(A) = 3$. 
Example in $M_7(\mathbb{C})$

$$A = \begin{bmatrix}
  a & c & 0 & \ldots & 0 \\
  b & a & c & \ddots & \vdots \\
  0 & \ddots & \ddots & \ddots & 0 \\
  \vdots & \ddots & b & a & c \\
  0 & \ldots & 0 & b & a
\end{bmatrix}$$

The numerical range of a $7 \times 7$ tri-diagonal Toeplitz matrix, with $a = 5 + 4i$, $b = -1 + i$, $c = -3$.

**Theorem**

$(C, R, S, S, 2014) \ k(A) = \left\lceil \frac{n}{2} \right\rceil$. 


